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# A MONOGRAPH ON TRUNCATED SEQUENTIAL TESTS

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## FOREWORD

This final report was prepared by Julian J. Bussgang and Nicholas Johnson of Signatron, Incorporated, Lexington, Massachusetts, under Contract AF30(602)-4356, project number DI 64-2. RADC project engineer is Haywood E. Webb (EMIA).

This book may not be distributed to CFSTI because the material in this report is planned in a Book on Sequential Defection.

This report has been reviewed and is approved.

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## ABSTRACT

This monograph presents an exposition of the subject of truncated sequential tests. Although the main intent of this report is tutorial in nature, some previously unpublished results have been included.

Special emphasis has been given to the following topics: a) the practical necessity for truncation, b) the effects of truncation on the performance of the test, c) the results of computer simulation, and d) rules of truncation and application to principles of design.

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# TABLE OF KEY SYMBOLS

| | |
|---|---|
| $a$ | Actual value of the parameter under test (signal-to-noise ratio) |
| $a_1$ | Minimum detectable value of <u>a</u> with performance $(\alpha, \beta)$ |
| $A$ | Upper boundary |
| ASN | Average sample number |
| $B$ | Lower boundary |
| $d_i$ | Decision to accept $H_i$ |
| $Ey$ | Expectation of random variable $y$ |
| $H_o$ | Null hypothesis $a=0$ |
| $H_1$ | Alternate hypothesis $a=a_1$ |
| ln | Natural logarithm |
| lnA | Upper threshold |
| lnB | Lower threshold |
| $L(a)$ | Probability of accepting $H_o$ when <u>a</u> is true |
| $L(a,N)$ | Probability of accepting $H_o$ when <u>a</u> is true in a truncated test |
| $m$ | Observation stage |
| $n$ | Terminal stage |
| $N$ | Truncation stage |
| OCF | Operating characteristic function |
| $p_a(n)$ | Total probability density of sample size when <u>a</u> is true |
| $p_a(n;N)$ | Total probability density of sample size when <u>a</u> is true in truncated test |
| $p_i(n\|d_j)$ | Probability density of sample size when $H_i$ is true and $H_j$ is accepted |
| $p_i(n\|d_j,N)$ | Probability density of sample size when $H_i$ is true and $H_j$ is accepted in a truncated test |

| | |
|---|---|
| $p(x_1,\ldots,x_m;a)$ | Probability density of sample $X_m$ when $\underline{a}$ is true |
| $p_0(X_m)$ | Probability density of $X_m$ given $a=0$ |
| $p_1(X_m)$ | Probability density of $X_m$ given $a=a_1$ |
| SPRT | Sequential probability ratio test |
| $t$ | Time of observation |
| $T$ | Truncation time |
| TST | Truncated sequential test |
| $x$ | Terminal threshold at truncation |
| $x_i$ | The $i^{th}$ observed value on the random variable under test |
| $X_m$ | The set of m successive observations $x_1,x_2,\ldots,x_m$ |
| $z_i$ | Logarithm of the probability ratio of $i^{th}$ observation |
| $Z_m$ | Logarithm of the probability ratio of first m observations |
| $\alpha$ | Probability of error of the first kind (false alarm) |
| $\alpha_T$ | Probability of error of the first kind in a truncated test |
| $\beta$ | Probability of error of the second kind (false dismissal) |
| $\beta_T$ | Probability of error of the second kind in a truncated test |

EVALUATION

This report is part of a series of tutorial monographs on various topics on system theory and information processing. The purpose of the series is to bring to the attention of electronic equipment designers, information resulting from recent research studies, in a tutorial form suitable for applications. The work was sponsored by the Laboratory Directors Discrectionary fund, under Task DI 64-2.

Sequential tests in electronic detection, have been considered for application to optimal detection problems (See for example RADC TDR 60-70A Applications of Decision Theory to Electronic Equipment." Today, some of those ideas have been applied to the design of practical, though sophisticated, radar equipments. Though a sequential detector is superior in performance than it's non-sequential counterpart, it is more complicated to impliment. It is only through understanding of principles, and the availability of performance calculations that one can determine if the extra sophistication justifies it in any particular application.

Though this report is primarily tutorial in nature, it does contain some previously unpublished results. The preparation of this report was motivated by the following conditions:

a) Expository material on the subject from an engineering point of view is sparse, though a chapter is included in the books, "Detection Theory" by Ivan Selin, and more recently""Signal Detection Theory" by Hancock and Wintz.

b) There are contributions from the engineering detection problem that are of use to statisticians.

It is hoped that the future will soon bring forth a rather full treatise in book form on this fascinating subject from an engineering point of view.

Haywood Webb Jr.
HAYWOOD WEBB
Project Engineer

# INTRODUCTION

The application of sequential tests to radar problems originated some ten years ago. At SIGNATRON we have studied extensively various truncated sequential tests. The current monograph is based on the results of a number of these studies.

Although the present monograph is intended to be tutorial in nature, it also contains some new previously unpublished material.

The authors are particularly grateful to their Air Force monitor, Mr. Haywood Webb of RADC for recognizing the need for the collection and formal presentation of this material. The authors also wish to acknowledge that the material presented in Section 6 of this monograph is based on the joint work of Julian J. Bussgang and Dr. Michael B. Marcus of the RAND Corporation which was published by the RAND Corporation.[1]

## 1. SEQUENTIAL TESTS AND RADAR DETECTION

The application of statistical tests of hypotheses to the radar detection problem is by now a well-known approach. As a result the practitioners of this field have developed a habit of intermingling the statistical language with the engineering one. The statistician, examining those of our writings on the subject that are concerned with engineering applications, frequently will fail to perceive the significant statistical contribution which may be contained in them. On the other hand the design engineer who wishes to extract significant information for his system will tend to be equally puzzled by the statistical terms and may fail to recognize the practical significance of the analytical results.

In order to overcome these difficulties this section of the monograph discusses the semantic equivalences of the engineering and statistical nomenclature used in what follows. This effort at translation, as if it were, should help to make the material of this monograph more accessible to workers in both fields. Moreover, it may also help to encourage the engineers to avail themselves of other material on sequential topics and vice versa.

The problem of deciding whether a target is present or absent can be regarded as a statistical test of two alternate hypotheses. Thus the design of the target detection system is the design of a statistical test. The observed variables are the received pulses. The detector is the device which constructs the test statistic. The test statistic is the function of observed variables, which is used to make decisions and is represented by the voltage at the output of the detector. The test procedure is the logic of operations on the received signal which have to be performed. The inequalities which have to be examined to reach a decision are tested by comparing the voltages at the output of the detector to suitable threshold voltages. The action following the decision that the target is present is

an "alarm". The action following the decision that the target is absent is a "dismissal". The probability $\alpha$ of accepting the alternate hypothesis, $H_1$ ("target present"), when the null hypothesis, $H_0$ ("target absent"), is true is the probability of error of the first kind. This probability in the radar context is the probability of false alarm. Similarly, the probability $\beta$ of accepting the null hypothesis $H_0$ ("target absent") when the alternate hypothesis $H_1$ ("target present") is true is the probability of error of the second kind. In the radar context this probability is the probability of false dismissal. If $\beta$ is the probability of false dismissal then $1 - \beta$ is the probability of detection, i.e., the probability of declaring target present when it indeed is present. The number of observations required to complete the test is known as the sample number and is, in effect, the number of radar pulses which must be received from the target to complete the detection process.

Since radar pulses are usually emitted at a uniform rate, the number of pulses to complete the detection process is directly proportional to the time required to complete the test and the notions of time and number of observations to complete the test are interchangeable. Since a likelihood ratio test on independent samples involves a sum of logarithms of likelihoods of individual observations the physical realization of a likelihood ratio detector involves an integrator. For a fixed sample size test, like a Neyman-Pearson test, this integrator has a fixed time constant. For sequential tests the time constant is variable.

## 2. SEQUENTIAL TESTS OF STATISTICAL HYPOTHESES

A sequential test is a statistical test of hypotheses where the amount of data which is examined is not specified in advance; the number of observations which are taken is a random variable. Such a procedure has the virtue that on the average the number of observations required is smaller than the fixed number of observations required in the fixed-sample procedure yielding the same probabilities of error. That is, with fixed-sample size hypothesis testing procedures, the number of observations to be taken is determined in advance and is a function of the probability of errors which will be tolerated. This fixed number of observations is then taken after which the value of the test statistic is computed and compared against a single threshold. One of the two hypotheses is accepted depending whether the test statistic does or does not exceed the threshold.

A sequential test, on the other hand, is performed by establishing two thresholds and successively taking observations until the test statistic which is recomputed with each new observation, exceeds the upper threshold or falls below the lower threshold. On the average, a sequential test which yields the same error probabilities will require fewer observations than the fixed sample procedure. Conversely, if the number of observations of the fixed sample procedure were chosen equal to the average number of observations of the sequential procedure, either one or both of the error probabilities of the sequential test would be smaller than the corresponding errors in the fixed sample test. Thus, either from the point of view of the number of observations, or from that of the error probabilities, the performance of the sequential procedure is superior to that of the conventional fixed sample size test.

The sacrifice which is made in exchange for this improved performance is the unpredictable number of observations required in the sequential procedure; i.e., although the average

number of observations can be expected to be smaller than the number of observations required in the fixed sample procedure, individual tests may require an extremely large number of observations relative to the average. In fact, some test may continue unresolved for such an extended period of time that the demand may arise that the testing procedure terminate and a decision be made as to the hypothesis which should be accepted. It is a recognition of this fact that leads us to a study of truncated sequential tests.

## 3. REVIEW OF RESULTS FOR UNTRUNCATED TESTS

The theory of sequential probability ratio tests (SPRT) as developed by Wald is presented in Ref. 2. In this section we do not develop the theory but present those general aspects of sequential analysis which will be necessary for a consideration of truncated sequential tests (TST).

Let us assume that we are interested in testing the null hypothesis $H_o$ that the value of the parameter $\underline{a}$ in the probability density function of a random variable x, $p(x;\underline{a})$, is zero against the alternative hypothesis $H_1$ that the value of the parameter is $\underline{a}_1$. According to the SPRT procedure, successive observations on the random variable x are made and after m observations (m=1,2,...) the logarithm of the likelihood ratio

$$Z_m = \ln \frac{p(x_1,x_2\ldots x_m;a_1)}{p(x_1,x_2\ldots x_m;0)} \qquad (3.1)$$

is computed and compared against two parallel thresholds (or boundaries) lnA and lnB. In statistical terminology $Z_m$ is the relevant test statistic. If $Z_m \geq \ln A$, the procedure is terminated by making decision $d_1$ to accept hypothesis $H_1$; if $Z_m \leq \ln B$, the procedure is terminated by making decision $d_o$ to accept hypothesis $H_o$; otherwise (when $\ln B<Z_m<\ln A$) another observation $x_{m+1}$ is taken and the testing procedure continues.

It can be shown that, neglecting excess over the boundaries,* the probability $\alpha$ of Type I error, i.e., rejecting $H_o$ when a = 0, and the probability $\beta$ of Type II error, i.e., accepting $H_o$ when $a=a_1$ are given respectively by

$$\alpha = \frac{1-B}{A-B} \quad \text{and} \quad \beta = \frac{AB-B}{A-B} \qquad (3.2)$$

*"excess over the boundaries" is the amount by which the value of $Z_m$ at termination stage n exceeds the boundary, i.e., $Z_n-\ln A$ or $\ln B-Z_n$.

Thus in order to construct a test with performance $(\alpha,\beta)$ we choose the thresholds $\ln A$ and $\ln B$, where

$$A = (1-\beta)/\alpha \qquad \text{and} \qquad B = \beta/(1-\alpha) \tag{3.3}$$

A pictorial representation of the testing procedure is given in Fig. 3-1. It will be noticed that the region of $Z_m$, $m>0$, is separated into three parts: accept $H_0$, accept $H_1$, and continue testing.

In what follows, we assume that all observations are independent so that

$$p(x_1,x_2,\ldots,x_m;a) = \prod_{i=1}^{m} p(x_i;a) \tag{3.4}$$

and hence

$$Z_m = \sum_{i=1}^{m} z_i \qquad m=1,\ldots,n \tag{3.5}$$

where

$$z_i = \ln \frac{p(x_i;a_1)}{p(x_i;0)} \tag{3.6}$$

We adopt the notation that n is a terminal stage of the standard SPRT and thus, neglecting the excess over the boundaries, $Z_n$ is the test statistic for a completed sample.

Since in radar problems a is, in general, associated with the signal-to-noise ratio, for the sake of simplicity and clarity in reference to errors of Type I and II we will use here the radar engineering terms: "false alarm" probability for the probability of Type I error and "false dismissal"

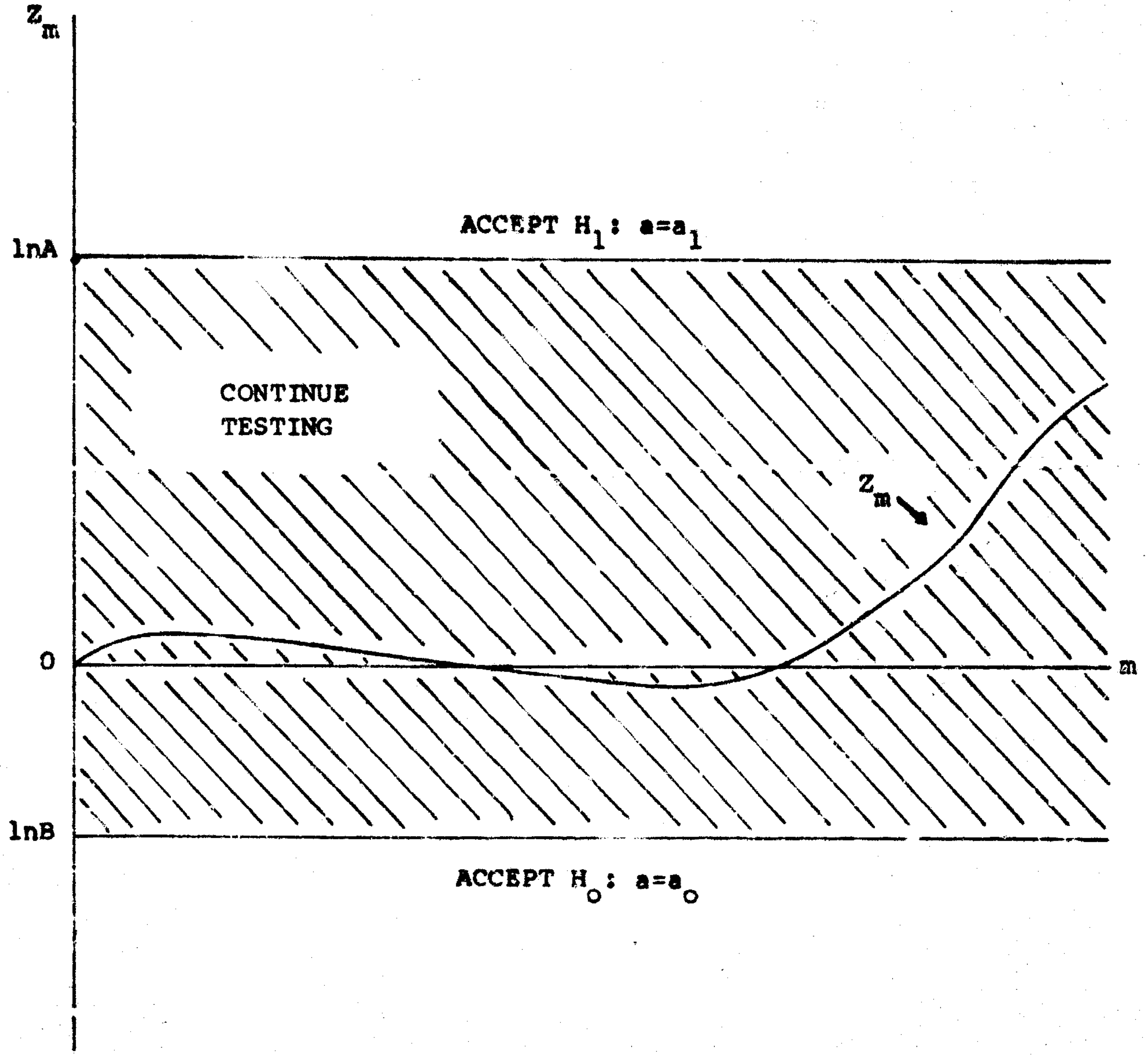


FIG. 3-1 PICTORIAL REPRESENTATION OF SEQUENTIAL DETECTION

probability for the probability of Type II error.

An important quantity in a sequential test is the average sample number (ASN), which is the number of observations needed on the average to arrive at a decision. The ASN is a function of the true value of the parameter $\underline{a}$, and for independent samples is given by

$$ASN = \frac{L(a)\ln B + [1-L(a)]\ln A}{Ez(a)} \qquad (3.7)$$

where L(a) is called the operating characteristic function (OCF) and is the probability of accepting hypothesis $H_o$ given that $\underline{a}$ is the true value of the parameter under test, and where Ez(a) is the expected value of z given $\underline{a}$:

$$Ez(a) = E[\ln \frac{p(x;a_1)}{p(x;0)} | a]$$

From the definition of $\alpha$ and $\beta$, it is evident that $L(0) = 1-\alpha$ and $L(a_1) = \beta$. Therefore, a feeling for the overall performance of a sequential test can be obtained simply by examining the OCF and ASN of the test.

It is a property of sequential tests that no other statistical test of hypothesis $H_o$ against $H_1$ can be constructed with a smaller average sample number, and yield as small an $\alpha$ and $\beta$ as a sequential test. Conversely no other test can achieve a smaller $\alpha$ and $\beta$ and also require on the average less observations than the sequential test.[6,8,10]

## 4. PRACTICAL NECESSITY FOR TRUNCATION

Although a proof by Stein[9] provides assurance that the sequential tests will terminate with probability 1, it is clear that some tests may last longer than can be tolerated. The test length is a random variable and long tests will occasionally occur even when the parameter a which is under test equals zero or exceeds the nominal value $a_1$ due to the non-zero variance of the test length. Furthermore, when the SNR which actually occurs has an intermediate value $0 < a < a_1$, the resulting ASN becomes itself very large thereby indicating that even, on the average, the test may take much longer than if a is exactly 0 or $a_1$. This last point can be understood by referring to Fig. 4-1 which illustrates typical ASN's. It follows that there will be occasions when the testing must be prematurely terminated and a decision reached on the basis of the already available data.

When situations arise when these very long tests can not be tolerated, it becomes necessary to modify the test procedure in order to accelerate the termination of the test. Sometimes it is even necessary to modify the test so that one can guarantee that the test will terminate prior to some specified stage $n = N$.

The necessity of truncation is especially evident in practical engineering problems where the urgency to terminate the testing procedure may increase with every succeeding sample. An example of such a situation is given by radar detection where a "hangup" on one particular target may allow other targets to pass by undetected or may cause a decrease of available radar response time.

This urgency can be stated in terms of a nonlinear increasing cost associated with successive samples. According to such a cost function succeeding samples become so costly that their relative value decreases rapidly thereby placing a

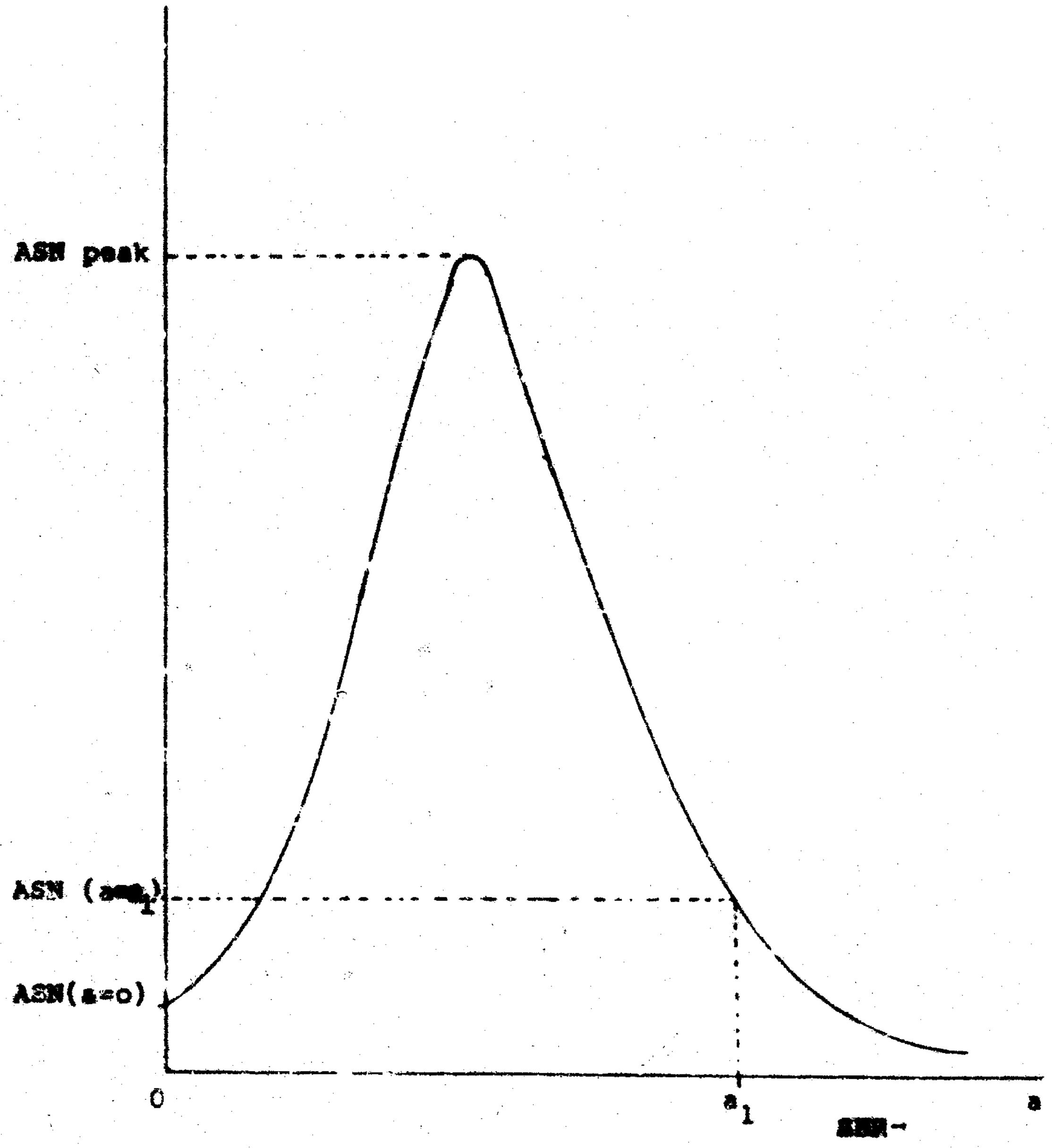


FIG. 4-1 A TYPICAL ASN

demand on the experimenter to end the testing. Several methods of accelerating or terminating the testing are presented in the next section.

## 5. RULES FOR TRUNCATION

There are several ways in which an accelerated or truncated sequential test can be constructed.

One way to achieve the desired change in the test procedure is to modify the parallel test boundaries shown in Fig. 3-1 to curve and meet at stage N. The stage N is called the truncation stage and the modified test a truncated test. Two manners of modifying the boundary deserve special attention. In the first case, no change in the test procedure is introduced until the truncation stage itself. In the other case the boundaries can be brought closer together with every stage to monotonically converge. In the former case, the truncation may be said to be abrupt and in the latter case gradual. It is, of course, also possible to let the boundaries asymptotically converge at infinity, guaranteeing that the test will accelerate but not necessarily terminate by a particular stage. Figure 5-1 illustrates some of the different ways of modifying the boundaries. When the boundaries do in fact meet at the truncation stage N, we specify at that stage only one rather than two thresholds so that there is no "defer-decision" region at that stage and the test must terminate.

It can be observed that from this point of view the parallel boundaries of the Wald test are a consequence of the uniform cost of additional observations independent of the stage number. On the other hand, the converging boundaries can be interpreted as a consequence of an ever increasing cost of additional observations so that the cost per observation is a non-linear monotonically increasing function of the stage number. The abrupt truncation becomes required when the cost of an additional observation becomes infinite.

When sequential tests are used in the context of a signal detection problem, it is sometimes possible that the circumstances of the problem permit the observer to control the energy of successive pulses. Suppose that the energy of successive pulses is gradually increased. Since the energy of successive pulses can be thought of as the cost of taking observations, the situation

in which the boundaries remain parallel but the energy increases is equivalent to the situation in which the energy of successive pulse remains the same but the boundaries converge. The procedure, based on gradually increasing the energy of successive pulses, implies feedback from the receiver to the transmitter carrying information whether or not the test has or has not yet terminated.

Modifications of the test procedure need not, of course, be restricted to the gradually converging boundaries, (or gradually increasing energy). It is possible, depending on the particular situation that the formulation of the problem leads to a multistage test of statistical hypotheses which does not entail monotonically converging boundaries (or increasing energy) but introduces more complicated boundary shapes. Some examples leading to such boundary shapes are discussed later in this section. Other topics relating to the modified sequential tests which are discussed include the analysis of performance of the modified tests and tests with variable energy.

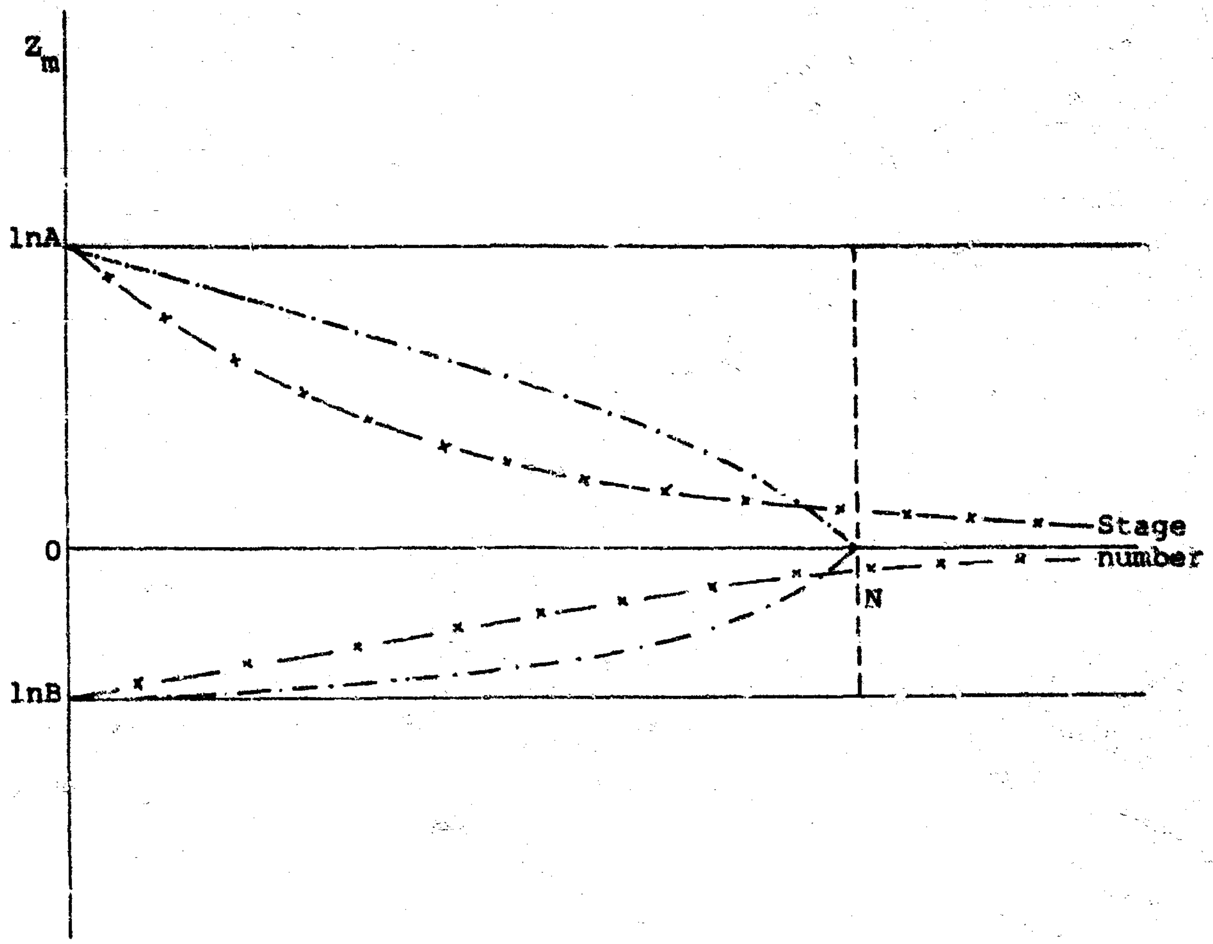


FIG. 5-1 DIFFERENT WAYS OF MODIFYING TEST BOUNDARIES

- - - - Abrupt truncation

—·—·— Gradual truncation

—x—x— Converging boundary

## 6. POLYNOMIAL APPROXIMATION TO MODIFIED BOUNDARIES

A sequential test with trapezoidal, sloping boundaries has been analyzed by Anderson[5] in the case of a random variable which is a Wiener process. More generally, we can consider a class of sequential test procedures that entail monotonically converging boundaries. A test procedure of this type can be called gradually truncated. A difficulty arises in the treatment of gradually truncated sequential tests in that the boundaries are themselves a function of the sample size, which is a random variable. In this section, we outline a method for obtaining approximately the Average Sample Number and the probability of accepting an alternate hypothesis. Our presentation is based on Bussgang and Marcus.[1]

Following Wald[2] consider the sequential test on a sequence $X_m=(x_1,x_2,\ldots,x_m)$ of discrete random variables. We assume that associated with the hypotheses $H_o$ and $H_1$ are the two alternate probability density functions $p_o(X_m)$ and $p_1(X_m)$ which govern the observations in the sample. The modified sequential test is performed as follows: with each new observation added to the sample, the likelihood ratio $p_1(X_m)/p_o(X_m)$ is formed. The process is continued as long as

$$e^{f_o(m)} < \frac{p_1(X_m)}{p_o(X_m)} < e^{f_1(m)} \qquad m=1,2,\ldots,n-1 \qquad (6.1)$$

and ceases at some stage n as soon as one side of the inequalies (6.1) is violated. The modification of the test consists of the fact that the boundaries are not constants but a function of m. For the sake of clarity, the terminal stage is denoted by n as distinguished from an arbitrary stage m; since the termination stage depends on the run of the sample, n is a random variable. Let the violation of the lower inequality be associated with the acceptance of $H_o$ and the violation of the upper inequality with the acceptance of $H_1$. Suppose the function $f_o(m)$ is monotonically

non-decreasing and the function $f_1(m)$ is monotonically non-increasing. Truncation occurs for the smallest value of m, N, at which $f_o(N)=f_1(N)$ since at that stage the inequality (6.1) must be violated. Notice that if the two hypotheses were equally likely and the costs of accepting incorrect decisions were equal, then it would be reasonable to require that at truncation $f_o(N)=f_1(N)=0$.

Let $E[f(n)|d_i,H_j]$ be the expectation of the function f of the terminal stage n given that the test terminates in the decision $d_i$ to accept the hypothesis $H_i$ and that the hypothesis $H_j$ is true; i=0,1; j=0,1. Bussgang and Marcus[1] (p.7) have shown that, neglecting the excess over the boundaries, the following two equalities hold for i=0,1 and j=1,0 respectively:

$$E[\exp f_i(n)|d_i,H_j] = P(d_i|H_i)/P(d_i|H_j) \qquad (6.2)$$

$$E[\exp -f_i(n)|d_i,H_i] = P(d_i|H_j)/P(d_i|H_i) \qquad (6.3)$$

and hence

$$E[\exp f_i(n)|d_i,H_j]\ E[\exp -f_i(n)|d_i,H_i] = 1 \quad \text{for } i\neq j \qquad (6.4)$$

For the case of constant boundaries $f_i(n)$=constant, i.e.,

$$f_1(n) = \ln A$$

$$f_0(n) = \ln B,$$

the result (6.4) becomes evident. Specifically, for i=1 and j=0, (6.2) and (6.3) imply

$$E[\exp f_1(n)|d_1,H_0] = (1-\beta)/\alpha \tag{6.5}$$

and

$$E[\exp -f_1(n)|d_1,H_1] = \alpha/(1-\beta) \tag{6.6}$$

To illustrate how these equalities can be used to evaluate approximately the performance of a sequential test consider the exponents specifying the boundaries that are of the form

$$f_0(n) = -\tilde{b}\left(1-\frac{n}{N}\right)^{r_0} \tag{6.7}$$

and

$$f_1(n) = \tilde{a}\left(1-\frac{n}{N}\right)^{r_1} \tag{6.8}$$

where $0<r_0,r_1\leq 1$ and $\tilde{a}$ and $\tilde{b}$ are positive. The graph of $f_1(n)$ is shown in Fig. 6-1. The graph of $f_0(n)$ is similar. In what follows assume independent observations and let the tilda sign (~) distinguish the quantities characterizing the modified test from the corresponding quantities in the Wald test. For the specified set of exponential boundaries the inequality (6.1) then becomes

$$-\tilde{b}\left(1-\frac{m}{N}\right)^{r_0} < \sum_{j=1}^{m} \ln\frac{p_1(x_j)}{p_0(x_j)} < \tilde{a}\left(1-\frac{m}{N}\right)^{r_1} \tag{6.9}$$

$$m=1,2,\ldots,n-1 < N$$

If $\tilde{a}/N$ and $\tilde{b}/N$ are small (i.e., N is large) the class of tests

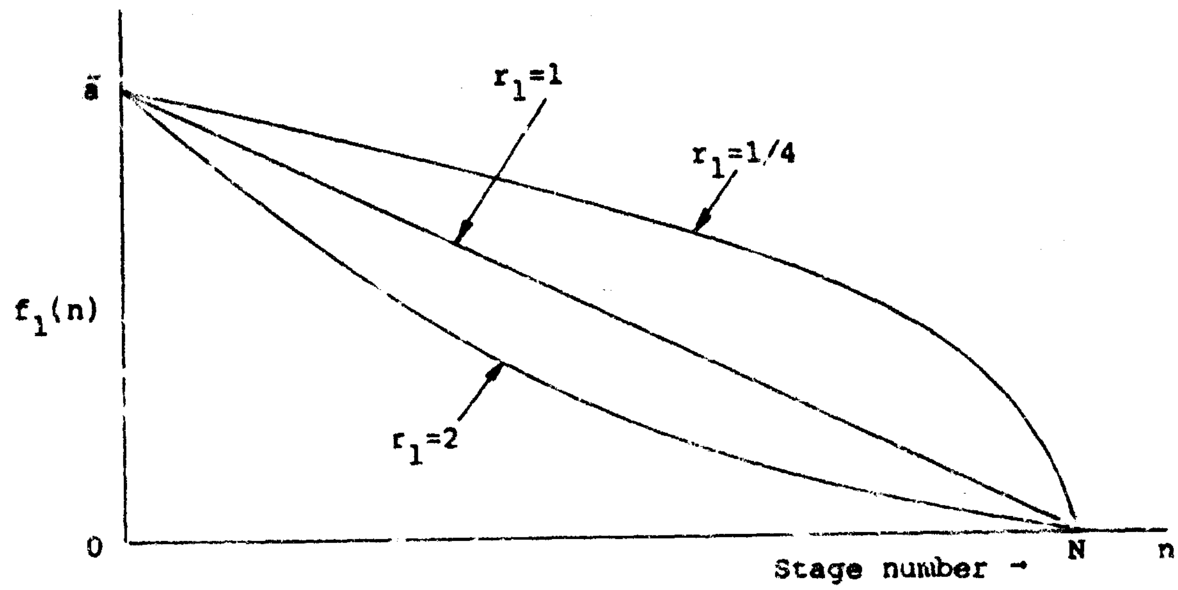


FIG. 6-1 THE UPPER BOUND $f_1(n)$ FOR A CLASS OF TRUNCATED TESTS

specified by (6.9) tends to the standard Wald tests with $a=\tilde{a}=\ln A$ and $\tilde{b} = b = -\ln B$. We consider the problem of finding the approximate ASN and the probabilities of error of the modified test $\tilde{\alpha}$ and $\tilde{\beta}$.

In order to simplify the resulting expressions assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are sufficiently small so that given $H_1$, the decision is $d_1$ most of the time; then

$$E(\tilde{n}|H_1) \approx E(\tilde{n}|d_1,H_1) \tag{6.10}$$

The following approximate expressions are obtained.

$$E(\tilde{n}|H_1) \approx \frac{\tilde{a}}{E(z|H_1) + r_1\tilde{a}/N} \tag{6.11}$$

and

$$\tilde{a} \approx e^{-\tilde{a}}\left(1 + \frac{r_1\tilde{a}^2}{NE(z|H_1) + r_1\tilde{a}}\right) \tag{6.12}$$

Similar expressions apply when the null hypothesis is true where $r_0$, $\tilde{b}$, $\tilde{\beta}$ replaces $r_1$, $\tilde{a}$, $\tilde{\alpha}$ and $E(n|d_0)$ replaces $E(n|d_1)$.

By keeping only the first moments of n, the boundaries are approximated by straight lines. We note that, in general, as the boundaries converge, excess over the boundaries will take place. The approximations in (6.11) and (6.12) ignore tests which terminate at the truncation stage N and are therefore meaningful only if circumstances of the problem are such that most tests terminate prior to the truncation stage.

## 7. ABRUPT TRUNCATION

In this section, we discuss an abruptly truncated sequential probability ratio test. The boundaries of the test remain parallel until the truncation stage N. At the truncation stage only one terminal threshold is used and the terminal decision must be made. In the interest of providing explicit results, the discussion in this Section will be specialized to apply to the test for the mean of a normal process. This corresponds to coherent detection of a known signal in white noise.

A truncated sequential procedure is illustrated in Figure 7-1. On the vertical axis is plotted the value of the test statistic $Z_n$ for (continuous) values of n on the horizontal axis. For each value of n the value of $Z_n$ is compared against the two thresholds lnA and lnB. If $Z_n$ exceeds the threshold lnA before $n=N$, the decision $d_1$ is made to accept the hypothesis $H_1$. If the value of $Z_n$ falls below the threshold lnB before $n=N$, the decision $d_o$ is made to accept the hypothesis $H_o$. If $Z_n$ remains between the two thresholds up to $n=N$, then at this stage no further samples are taken and the value of $Z_n$ is compared with the terminal threshold x. Hypothesis $H_1$ is accepted if $Z_n>x$, otherwise hypothesis $H_o$ is accepted.

The four labeled paths in Figure 7-1 show the four possible ways in which a detection procedure can terminate in a truncated test. Decision $d_1$ is made when either the test statistic $Z_n$ exceeds lnA or remains within the two parallel boundaries and is greater than the terminal threshold x at the truncation point N. These two possibilities are shown by path (1) and (2). Decision $d_o$ is made whenever $Z_n$ crosses the lower boundary lnB, or remains in the deferred decision region until the test is truncated and the test statistic is then below x. These two possibilities are shown by paths (3) and (4).

The mathematical simplicity of the expressions (3.2) and (3.7) which specify the performance of a sequential test is lost to a large extent when truncated sequential tests (TST) are considered. The difficulties center about the fact that now we must

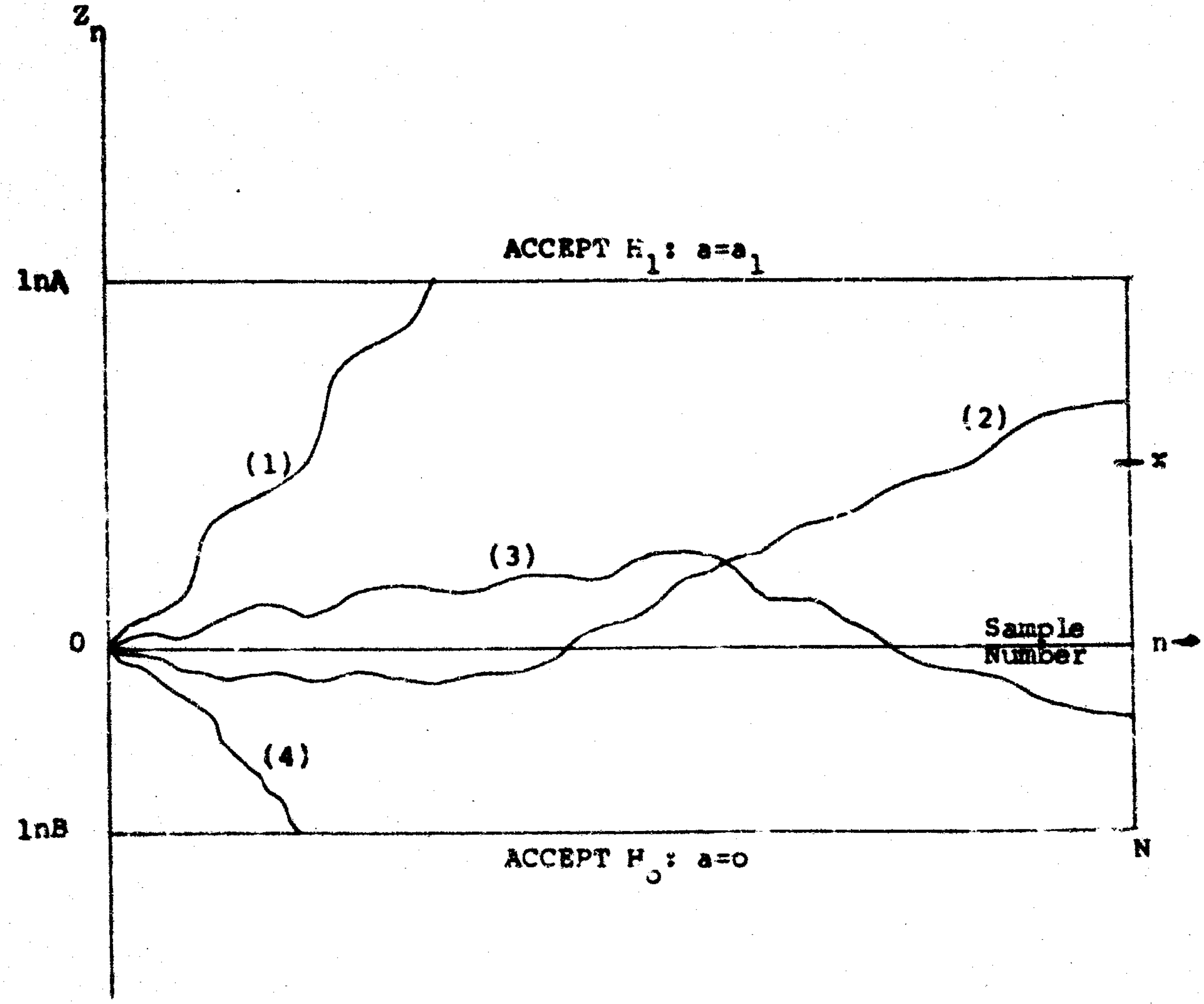


FIG. 7-1 PICTORIAL REPRESENTATION OF TRUNCATED SEQUENTIAL TESTS

consider separately the outcome of tests which terminate before truncation and those that terminate at truncation. There is no simple transition from the analysis of the untruncated procedure to the truncated case. Let the probabilities of error of a TST be $\alpha_T$ and $\beta_T$.

A truncated test is a compromise between an entirely sequential test and a fixed test. It is an attempt to reconcile the good features of both of them: the sequential feature of examining observations as they accumulate and the fixed test feature of guaranteeing completion within a specified sample size. It is clear that modification of the test boundaries changes both the probabilities of error and the distribution of the sample size. For example, if in Fig. 7-1 the threshold lnA and lnΓ are set to yield errors ($\alpha,\beta$) in the untruncated procedure, the introduction of a truncation point at n=N will decrease the ASN but increase the values of $\alpha$ and/or $\beta$. The net effect is a loss of performance. That is if we could adjust the thresholds of the truncated test so that $\alpha_T=\alpha$ and $\beta_T=\beta$ then the truncated procedure would have a larger ASN than the untruncated test. Conversely if we could adjust the thresholds of the truncated test so that the ASN of the truncated test were equal to the ASN of the untruncated test for $0\leq a\leq a_1$, then either or both $\alpha_T$ and $\beta_T$ would be greater than the corresponding values of the untruncated test with the unadjusted boundaries. It should be recognized that in the truncated procedure the choice of the location of the terminal threshold x influences the values of $\alpha_T$ and $\beta_T$. This influence decreases as the truncation stage increases. Thus for very large values of N (large in comparison to the ASN), the values of $\alpha_T$ and $\beta_T$ are not sensitive to the value of x; for small values of N the values of $\alpha_T$ and $\beta_T$ are very sensitive to the position of the terminal threshold x.

In a SPRT the probabilities of errors $\alpha$ and $\beta$ are not a function of the parameter $a_1$, but are dependent only on the setting of the thresholds lnA and lnB. The ASN, however, is dependent both on the values of $a_1$ and $\underline{a}$. In fact, if the variable under test is the mean of a Gauss distribution, the quantity of interest is not ASN alone but $a_1^2$ASN, i.e., the

"average energy" of the test. Although the ASN varies not only with $\underline{a}$ but also with $a_1$, the quantity $a_1^2$ASN varies only with $\underline{a}$. In the fixed sample test for the mean of the Gauss distribution, we recall[3] the error probabilities are a function of the "total energy" $a_1^2 n_f$ where $n_f$ is the total (and fixed) number of samples; the larger $a_1$ the smaller the value of the $n_f$. Consequently for the case of a TST, a fixed value of N becomes relatively large in comparison with the ASN as $a_1$ increases. The larger N is in comparison with the ASN, the more the TST resembles a SPRT test and correspondingly $\alpha_T$ and $\beta_T$ approach $\alpha$ and $\beta$. Thus for a TST the value of $a_1$ influences the probabilities of error. Later it is shown that the performance parameters of a TST are in fact universal functions of the truncation energy in the test for the mean of white noise.

As stated previously to any TST there will be a net loss in the performance over the untruncated SPRT. In fact, if we attempt to maintain $\alpha_T=\alpha$ and $\beta_T=\beta$ as $a_1^2N$ decreases, the separation between thresholds lnA and lnB must be increased accordingly, until in the limit lnA=$\infty$, lnB=$-\infty$, $a_1^2$ASN=$a_1^2$N and the resulting TST has become a fixed sample test. At this point the net savings in energy of a TST over a fixed sample test is, of course, zero.

## 7.1 Explicit Results for the Test for the Mean of White Noise

In general, the expressions for $\alpha_T$ and $\beta_T$ as a function of the boundaries are difficult to obtain. However, when the logarithm of the likelihood ratio can be represented by a Wiener process, or the sum of a Wiener process and a deterministic function of time, explicit results are available from the work of T. W. Anderson.[5] The logarithm of the likelihood ratio, i.e., the test statistic can be approximately represented in this way when the sequential test (SPRT) is for the mean of a normal variable on successive independent samples. The derivation of these results is given in Appendix A. Here we give approximations which are valid for a variety of applications.

The samples under test follow the density function

$$f(x_i,a) = \frac{1}{\sqrt{2\pi}} \exp-(x_i-a)^2/2 \tag{7.1}$$

The null hypothesis is $a=0$ and the alternate hypothesis is $a=a_1$. We let $t$ be the time variable so that, if $\Delta T$ is the time between observations and $m$ the number of observations, the elapsed time then is $t=m\Delta T$. Let $T$ be the truncation time so that $T=N\Delta T$. For a test represented in Fig. 7-1, we then have the following approximate results:

$$\alpha_T \doteq 1 - \Phi\left(\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2}\right) + \frac{1}{A}\Phi\left(\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2} - \frac{2\ln A}{a_1\sqrt{T}}\right)$$

$$- \frac{B}{A}\Phi\left(\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2} - \frac{2(\ln A-\ln B)}{a_1\sqrt{T}}\right)$$

$$- \frac{1}{B}\Phi\left(-\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2} + \frac{2\ln B}{a_1\sqrt{T}}\right) + \frac{A}{B}\Phi\left(-\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2} - \frac{2(\ln A-\ln B)}{a_1\sqrt{T}}\right) \tag{7.2}$$

and

$$\beta_T \doteq \Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2}\right) + B\Phi\left(-\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2} + \frac{2\ln B}{a_1\sqrt{T}}\right)$$

$$- \frac{B}{A}\Phi\left(-\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2} - \frac{2(\ln A-\ln B)}{a_1\sqrt{T}}\right) - A\Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2} - \frac{2\ln A}{a_1\sqrt{T}}\right)$$

$$+ \frac{A}{B}\Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2} - \frac{2(\ln A-\ln B)}{a_1\sqrt{T}}\right) \tag{7.3}$$

These last two equations demonstrate that the probabilities of error of this truncated test are universal functions of the "truncation energy" $a_1^2T$. It can be determined directly from eqs (7.2) and (7.3) that if we let $\ln A=\infty$ and $\ln B=-\infty$ the resulting equations for $\alpha_T$ and $\beta_T$ are exactly those that express the probabilities of error for a fixed sample test with energy $a_1^2T_f$ and threshold x, namely,

$$\alpha_f = 1 - \Phi\left(\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T_f}}{2}\right) \tag{7.4}$$

$$\beta_f = \Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T_f}}{2}\right) \tag{7.5}$$

From the expressions (7.4) and (7.5) it also follows that in order to achieve performance $\alpha_f$ and $\beta_f$, the (terminal) threshold x must be set at

$$x = \frac{1}{2}\left\{[\Phi^{-1}(\alpha_f)]^2 - [\Phi^{-1}(\beta_f)]^2\right\} \tag{7.6}$$

and the test must have energy

$$a_1^2T_f = [\Phi^{-1}(\alpha_f) + \Phi^{-1}(\beta_f)]^2 \tag{7.7}$$

The last two relations are the known results for fixed sample hypothesis tests.

## 7.2 A Truncated Sequential Test with One Threshold at Each Stage

We see from (7.2) and (7.3) that $\alpha_T$ and $\beta_T$ are a universal function of $a_1^2T$ and depend on the three quantities lnA, lnB, and

n. There are therefore three quantities which can be adjusted to satisfy the conditions that $\alpha_T$ and $\beta_T$ have specified values. In practical problems a third condition to uniquely specify those three quantities for a given set ($\alpha_T$, $\beta_T$) must be judiciously chosen.[3] For example, in the case of sequential detection of a radar return, the probability of an echo from space is so small that it is very unlikely that the test will terminate by the test statistic intersecting the "target present" threshold lnA. Therefore, most of the time the test lengths will be controlled by the location of the lower threshold lnB; the closer this threshold to zero, the more likely is the test statistic to cross it early and the shorter the test length. Thus for any value of $a_1^2T$ we can minimize the dismissal test length by raising the lower threshold as close to zero as possible while simultaneously lowering the value of x and raising lnA in order to maintain the desired values of $\alpha_T$ and $\beta_T$. Under those conditions, the value lnA=∞ is the value of the upper threshold which leads to the smallest value of -lnB. Since no alarm could occur prior to the terminal stage, the probability of false alarm $\alpha_T$ would then be determined solely by the terminal threshold x. Under these circumstances only one threshold exists at any stage. For such a single boundary test (Fig. 7-2) the general expressions (7.2) and (7.3) simplify considerably and for the target-absent and target-present cases, we obtain:

$$\lim_{A\to\infty} \alpha_T = 1 - \Phi\left(\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2}\right) - \frac{1}{B}\Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2} + \frac{2\ln B}{a_1\sqrt{T}}\right) \tag{7.8}$$

and

$$\lim_{A\to\infty} \beta_T = \Phi\left(\frac{x}{a_1\sqrt{T}} - \frac{a_1\sqrt{T}}{2}\right) + B\Phi\left(-\frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}}{2} + \frac{2\ln B}{a_1\sqrt{T}}\right) \tag{7.9}$$

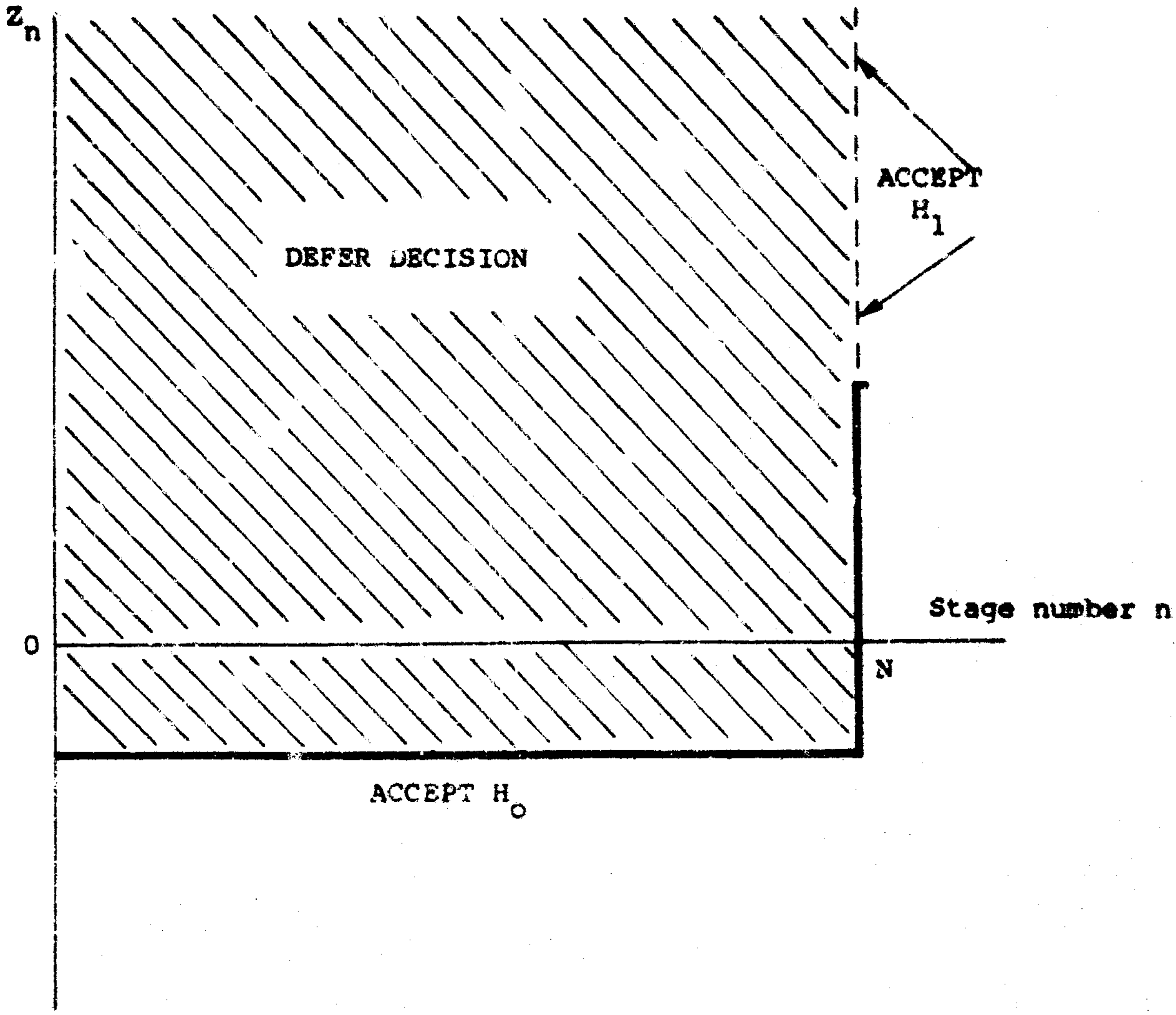


FIG. 7 2 TRUNCATED SEQUENTIAL TEST WITH ONE BOUNDARY PER STAGE

Furthermore, under the same condition and using eq. (B.1) in Appendix ( B ) the density of t can be shown to be

$$p(t) = -\frac{\ln B}{a_1^3 t^{3/2}} \; \phi\left(\frac{a_1\sqrt{t}}{2} + \frac{\ln B}{a_1\sqrt{T}}\right) \quad \text{for } t<T \tag{7.10}$$

If we relaxed the controls on the test and were content with tolerating a larger value for the probability of false alarm $\alpha_T$ than given by (7.8), but maintain $\beta_T$, we could achieve this by lowering x and correspondingly raising lnB while maintaining $\ln A=\infty$. The ASN of course decreases, reflecting the increase in $\alpha_T$. As we continue to relax our requirement on $\alpha_T$, and lower x, the terminal threshold x would eventually coincide with lnB. For a continuous process lowering x below lnB is meaningless. Thus even for this minimum value of x, the resulting test may have a value of $\alpha_T$ smaller than the required $\alpha$. Exact equality could then be achieved only by lowering the threshold lnA. This in turn would have a tendency to further decrease the ASN of the test, but as we reintroduce the upper threshold, the important problem of a non-uniqueness of threshold setting which yields the desired probability of error arises.

## 7.3 Interaction Between Thresholds, ASN, and Probabilities of Error

A TST of Fig. 7-1 can be specified when the three threshold A,B and x are given. Usually the test requirements are stated in terms of the probabilities of false alarm ($\alpha_T$) and false dismissal ($\beta_T$). A third condition that could be set in determining the three thresholds which will result in a TST with specified controls is to require that the ASN of theTST be a minimum. Of course, the ASN is a function of the value of the parameter a, and as such we can require minimization of ASN for a particular value of a or for a in a specified range. The exact expression for the ASN is given in the Appendix (A.3). For most applications the first term of the infinite sum (A.8) yields sufficient accuracy.

An example of the effects on $\alpha_T$ and $\beta_T$ as a result of varying the thresholds is given in Figure 7-3. In the first column we number the case under consideration; in the next column we consider raising (+) or lowering (-) the threshold given in the heading; the effects of this change on the thresholds are given in the last section, where "+" or "-" signify an increase or decrease in the values of $\alpha_T$, $\beta_T$. Initially thresholds lnA, lnB, and x yield performance of $\alpha_T$, $\beta_T$ and ASN. The results of each case is compared with this initial situation. For example, in Case 1, raising lnA will decrease the probability of false alarm; to offset this a decrease in the terminal threshold x will increase $\alpha_T$ and decrease $\beta_T$. The net effect on these probabilities of error could be zero, but the ASN would have been increased due to the raising of the upper threshold lnA.

Not all possible cases are illustrated in Fig. 7-3, since in some cases it is not clear what would be the net effect on the ASN for certain adjustments of the threshold. For example, if lnB and lnA were increased simultaneously the resulting probability of false alarm would decrease whereas the probability of false dismissal would increase. A lowering of the terminal threshold could possibly compensate for this change in the error probabilities. However there is no assurance that the ASN is uniformly larger or smaller than in the case before the thresholds were adjusted. If <u>a</u> is close to zero, then the new ASN will be smaller than in the previous case; if <u>a</u> is close to $a_1$, the converse is true. Thus whether the value of ASN increases or decreases in such a case depends on the true value of the parameter <u>a</u>.

## 7.4 Setting the Terminal Threshold

It is evident that there are at least two ways of approaching the problem of truncated sequential tests. The first is to determine the adjustment on the three thresholds in order to arrive at the desired values of $\alpha$ and $\beta$ according to some criterion on the ASN. The second is to fix the parallel thresholds and determine the position of the terminal threshold in some

| Case No. | Change In lnA | lnB | x | Effect On $\alpha_T$ | $\beta_T$ | Overall Effect On ASN |
|---|---|---|---|---|---|---|
| 1 | + | | | − | + | + |
| | | | − | + | − | |
| 2 | | − | | + | − | + |
| | | | + | − | + | |
| 3 | | + | | − | + | − |
| | | | − | + | − | |
| 4 | − | | | + | − | − |
| | | + | | − | + | |
| | | | + | − | + | |

Fig. 7-3 Possible Adjustments of Thresholds of a TST of Fig. 7-1 to Maintain Given $(\alpha_T, \beta_T)$ with Resulting Change in the ASN

("+" indicates that the value has been increased and "-" that it has been decreased)

desirable way. The first problem has been considered above to some extent. We now consider the second problem.

Wald[2] recognized the necessity of a terminal decision in a truncated test but was only able to give an intuitive answer to the question of where the threshold should be set. He felt that "a simple and reasonable rule for truncation..." was to set the terminal threshold at zero and accept or reject the null hypothesis depending on whether or not the test statistic at truncation is positive or negative. This rule seems reasonable mainly due to the fact that Wald, as most statisticians, was concerned with values of $\alpha$ and $\beta$ which are about equal, as a result of which zero is approximately halfway between lnA and lnB. In many scientific or engineering applications, however, $\alpha$ and $\beta$ may differ by several orders of magnitude. For example, in radar sequential detection $\alpha$ is typically in the order of $10^{-4}$ to $10^{-10}$ whereas $\beta$ is typically in the order of $10^{-1}$ to $10^{-3}$. Thus the choice of zero for the terminal threshold in such a case will have a tendency to disproportionally favor the type of error over the other, that is, if many tests are resolved at the truncation stage, such a choice may have a tendency to drastically change the magnitude of the errors for which the test was designed.

There is no unique criterion from which the value of the terminal threshold can be determined when a parallel boundary test is truncated. Ideally we would want x to be chosen such that the decision at truncation would be exactly the one which would be made had the testing continued. This ideal situation is unachievable for it would imply that a TST could perform as well as the optimum test.

One possible solution is to set

$$x = \frac{(\ln A + \ln B)}{2}$$

so that the terminal threshold is halfway between the parallel thresholds lnA and lnB. This rule essentially directs the experimenter to accept the hypothesis that corresponds to the

boundary to which $Z_T$ is closest. This approximation is of course very useful in engineering applications where a simplified technique is of greater importance than exact results.

## 8. DENSITY DISTRIBUTIONS OF TST

### 8.1 Total Distribution Functions

There are several density functions of the sample size which can be studied, several of which are related.* Probably the most important distribution is the density of the sample size which is not conditioned on the particular terminal decision. This distribution gives the density $p_a(n)$ of the number of samples n required to terminate with either decision when a is the true value of the parameter. The density $p_a(n)$ is the total density function conditional on a. It follows that the ASN of the test is given by

$$ASN = \int_0^\infty n p_a(n)\, dn \tag{8.1}$$

Now if the test is truncated at n=N and the thresholds are left unchanged, the new density function $p_a(n;N)$ is unchanged up to n=N, at which point it takes the form of a delta function of strength $\int_N^\infty p_a(n)dn$. Thus we have

$$p_a(n;N) = \begin{cases} p_a(n) + \delta(n-N) \int_N^\infty p_a(n)\, dn & \text{for } n \leqq N \\ 0 & n > N \end{cases} \tag{8.2}$$

and the new ASN takes the form

$$ASN = \int_0^N n p_a(n)\, dn + N \int_N^\infty p_a(n)\, dn \tag{8.3}$$

---

*Although the sample size, n, is a discrete variable, the distribution of n can be approximated by a continuous distribution when the expected excess over the boundaries is small and the step size $z_i$ is small relative to lnA and |lnB|.

By definition, an abruptly truncated test can last no longer than N observations, and it can be shown that on the average, it must take less observations than the untruncated test with the same parallel boundaries. It follows from (8.1) and (8.3) that the difference

$$ASN - ASN_N = \int_N^{\infty} (n-N)\ p(n)\ dn \tag{8.4}$$

and is always positive, showing that $ASN > ASN_N$. Graphically, the density function and the distribution function of the sample number for an untruncated and a truncated test are shown in Figs 8-1 and 8-2. Several approximate expressions for $p_a(n)$ are available; an exact expression for a Wiener process, as a function of the thresholds and the value of a is given in the Appendix. The expression for the $ASN_N$ can be obtained by means of (8.3) and is also given in the Appendix A.3.

## 8.2 Conditional Distribution Functions

Other distributions of interest are given by the condional distributions of the sample size. These are distributions of the sample size given that decision $d_i$ is made, i=0, 1 when a is the true value of the parameter, and will be denoted by $p_a(n|d_i)$. It can be shown[4] that in the untruncated case these conditional densities are related to the total density by the expression

$$p_a(n) = L(a)\ p_a(n|d_o) + [1-L(a)]\ p_a(n|d_1) \tag{8.5}$$

We first develop the corresponding expression for a TST, and point out some interesting consequences. We will then find various relationships between conditional and unconditional density functions.

Let $\mu_a(n|d_i;N)$ be the probability measure of the set of those tests which lead to decision $d_i$ at stage $n \leq N$ when $\underline{a}$ is the true value of the parameter. The probability that the TST terminates at the $n^{th}$ stage with either decision given that truncation is at $n=N$, is given by

$$p_a(n;N) = \mu_a(n|d_o;N) + \mu_a(n|d_1;N) \tag{8.6}$$

It is important to point out here that $\mu_a(n|d_i;N)$ for $i=0,1$ is <u>not</u> a function of N when $n<N$. That is, tests which have terminated at $n=m<N$ are not influenced by the value of N; only those tests terminating at $n=N$ are affected by the truncation stage. Of course, at $n=N$, the value of the test statistic is compared against the terminal threshold x in order to arrive at a decision. The location of this threshold will determine which tests at $n=N$ will declare decision $d_i$.

Now the OCF, which is the probability of accepting $H_o$, can be written as

$$L(a,N) = \sum_{n=1}^{n=N} \mu_a(n|d_o;N) \tag{8.7}$$

from which it follows that

$$1 - L(a,N) = \sum_{n=1}^{n=N} \mu_a(n|d_1;N) \tag{8.8}$$

These expressions can be written in the combined form

$$L_i(a,N) \equiv [(1-2L(a,N))]\, i + L(a,N) = \sum_{n=1}^{N} \mu_a(n|d_i,N), \quad i=0,1 \tag{8.9}$$

where we define $L_i(a,N)$ as the probability of accepting hypothesis $H_i$ for a test truncated at $n=N$. Also, the conditional probability function $p_a(n|d_i;N)$ can be written as

$$p_a(n|d_i;N) = \frac{\mu_a(n|d_i;N)}{\sum_{n=1}^{n=N} \mu_a(n|d_i;N)}\,, \quad n \leq N \tag{8.10}$$

This is the probability that the TST will terminate at stage n given that the decision is $d_i$ and that $\underline{a}$ is the true value of the parameter.

It is also important to point out here that although $\mu_a(n|d_i;N)$ is independent of N for $n<N$, $p_a(n|d_i;N)$ __is__ dependent on the value of N for $n<N$, as well as for $n=N$; $L(a,N)$ of course, is also dependent on N.

Using (8.7), (8.8), and (8.10) with $i=0$, and $i=1$ in (8.6) we get

$$p_a(n;N) = L(a,N)\, p_a(n|d_0;N) + [(1-L(a,N)]\, p_a(n|d_1;N) \tag{8.11}$$

For the usual two cases of interest we can write (8.11) as

$$p_0(n;N) = (1-\alpha_T)\, p_0(n|d_0;N) + \alpha_T\, p_0(n|d_1;N) \tag{8.12a}$$

for signal absent,

and, for signal present,

$$p_1(n;N) = \beta_T\, p_1(n|d_o;N) + (1-\beta_T)\, p_1(n|d_1;N) \qquad (8.12b)$$

Thus we have arrived at the analogue of (8.5) for TST and to the interesting conclusion that although each of the two factors in each term on the right-hand side of (8.11) varies with the value of N, each term (or the product of these two factors) is independent of N for n<N.

Once we have the probability function of the sample size of an untruncated test conditional on the decision $d_i$ being made, we may then wish to determine the corresponding conditional probability functions of a TST.

Let us assume, then, that we are given $p_a(n|d_i)$, the conditional density of the sample size n for an untruncated sequential procedure when a is the true value of the parameter given that $d_i$, i=0,1. is the decision made. We wish to determine $p_a(n|d_i;N)$ for $n\leq N$, where N is the truncation stage.

Using the same notation as above we have that the measure of those tests which terminate with decision $d_i$ at stage n in the untruncated procedure is

$$\mu_a(n|d_i) = p_a(n|d_i)\, L_i(a) \qquad (8.13)$$

where $L_i(a)=\sum_{n=1}^{\infty}\mu_a(n|d_i)$ is the probability of making decision $d_i$ when a is the value of the parameter. In the truncated procedure the corresponding relationship is given by (8.9) where $\mu_a(n|d_i;N)=\mu_a(n|d_i)$, when n<N; and for n=N:

$$\mu_a(N|d_i;N) = L_i(a,N) - \sum_{k=1}^{N-1} \mu_a(k|d_i;N)$$

or

$$\mu_a(N|d_i;N) = L_i(a,N) - L_i(a)\sum_{k=1}^{N-1} p_a(k|d_i) \qquad (8.14)$$

Now, using (8.9) and (8.10) we can write

$$p_a(n|d_i;N) = \frac{\mu_a(n|d_i;N)}{L_i(a,N)} \qquad (8.15)$$

or substituting for $\mu_a(n|d_i;N)$, we get

$$p_a(n|d_i;N) = \begin{cases} \dfrac{L_i(a)}{L_i(a,N)}\, p_a(n|d_i), & n<N \\ \\ 1 - \dfrac{L_i(a)}{L_i(a,N)} \displaystyle\sum_{k=1}^{N-1} p_a(k|d_i), & n=N \end{cases} \qquad (8.16)$$

As examples of this last expression we can write, for signal absent,

$$p_o(n|d_1;N) = \begin{cases} \dfrac{\alpha}{\alpha_T}\, p_o(n|d_1), & n<N \\ \\ 1 - \dfrac{\alpha}{\alpha_T} \displaystyle\sum_{k=1}^{T-1} p_o(k|d_1), & n=N \end{cases} \qquad (8.17a)$$

and, for signal present,

$$p_1(n|d_o;N) = \begin{cases} \dfrac{\beta}{\beta_T}\, p_1(n|d_o), & n<N \\ \\ 1 - \dfrac{\beta}{\beta_T} \displaystyle\sum_{k=1}^{N-1} p_1(k|d_o), & n=N \end{cases} \qquad (8.17b)$$

We can make certain observations at this point: It is not sufficient to know $p_a(n|d_i)$ in order to obtain $p_a(n|d_i;N)$; certainly the value of $L_i(a)$ and $L_i(a,N)$ are needed. However, $L_i(a,N)$ can be easily tabulated as a function of N in contrast to $p_a(n|d_i;N)$. Thus we have reduced the problem of determining values of $p_a(n|d_i;N)$ as a function of n and N to the simpler problem of tabulating $L_i(a,N)$ as a function of N and using (8.17) to obtain the desired distribution function of the sample size. In particular, if the desired quantities are to be obtained from experimental results then $L_i(a,N)$ can be easily obtained for each value of N in the course of determining $p_a(n|d_i)$ and $L_i(a)$. The quantity $p_a(n|d_i;N)$, on the other hand can not be obtained directly because of the dependence of $p_a(n|d_i;N)$ on N for <u>each</u> $n<N$, as was noted earlier. Thus, it is not practical to tabulate $p_a(n|d_i;N)$ for several values of N even though the expression could be calculated at each stage of the sequential procedure. The expression (8.17) developed above is recommended when selected values of $p_a(n|d_i;N)$ are desired.

The relationships between the various conditional and unconditional distributions for TST can be obtained by starting with some known relationships and the ones derived in the preceding section. A summary of these relationships is offered in Table 1. Some of the expressions are equivalent although presented in different form. These are listed in order to facilitate their use depending on a user's individual need.

Certain comments can be made with regard to the expressions given in Table 1.

(a) The introduction of a truncation stage N implies the use of a terminal threshold x which determines the decision to be made at this stage. Thus quantities such as $\alpha_T$ and $\beta_T$ are in fact a function of this terminal threshold although its functional dependence has not been explicitly indicated.

(b) The Theorem of Bussgang-Marcus which establishes the equality between the density functions given in (3) of Table 1 does not hold for the case of TST.[1,6] These

latter expressions are given in (6) of Table 1.

(c) The expression $p_o(n|d_o;N)$ in (6) depends on N only through $\alpha_T$. Similar statements hold for the other density functions in (6) of Table 1.

(d) Since, for example, $\alpha_T p_o(n|d_1,N)$ does not depend on N for n<N we can write

$$\alpha_T \, p_o(n|d_1;N) = \alpha \, p_o(n|d_1) \qquad n<N$$

(e) For continuous testing in a process with independent increments we would replace the summations of form $\sum_{k=1}^{N-1}$ by the integrals $\int_0^T dt$.

**TABLE 1**

**SUMMARY OF RELATIONSHIPS**

1
$$\begin{cases} p_o(n) = (1-\alpha)\ p_o(n|d_o) + \alpha\ p_o(n|d_1) \\ p_1(n) = \beta\ p_1(n|d_o) + (1-\beta)\ p_1(n|d_1) \end{cases}$$

2
$$\begin{cases} p_o(n|N) = (1-\alpha_T)\ p_o(n|d_o;N) + \alpha_T\ p_o(n|d_1;N) \\ p_1(n|N) = \beta_T\ p_1(n|d_o;N) + (1-\beta_T) p_1(n|d_1;N) \end{cases}$$

3
$$\begin{cases} p_o(n|d_o) = p_1(n|d_o) & \text{Theorem of Bussgang-Marcus} \\ p_o(n|d_1) = p_1(n|d_1) \end{cases}$$

4
$$\begin{cases} p_o(n|d_o) = \dfrac{(1-\beta)\ p_o(n) - \alpha\ p_1(n)}{(1-\alpha-\beta)} = p_1(n|d_o) \\ p_1(n|d_1) = \dfrac{(1-\alpha)\ p_1(n) - \beta\ p_o(n)}{(1-\alpha-\beta)} = p_o(n|d_1) \end{cases}$$

5
$$p_o(n|d_1;N) = \begin{cases} \dfrac{\alpha}{\alpha_T}\ p_o(n|d_1) & n < N \\ 1 - \dfrac{\alpha}{\alpha_T} \displaystyle\sum_{k=1}^{N-1} p_o(k|d_1) & n = N \end{cases}$$

$$p_1(n|d_o;N) = \begin{cases} \dfrac{\beta}{\beta_T}\ p_1(n|d_o) & n < N \\ 1 - \dfrac{\beta}{\beta_T} \displaystyle\sum_{k=1}^{N-1} p_1(k|d_o) & n = N \end{cases}$$

5 (cont.)

$$p_o(n|d_o;N) = \begin{cases} \dfrac{1-\alpha}{1-\alpha_T}\, p_o(n|d_o) & n < N \\[2ex] 1 - \dfrac{1-\alpha}{1-\alpha_T} \displaystyle\sum_{k=1}^{N-1} p_o(k|d_o) & n = N \end{cases}$$

$$p_1(n|d_1;N) = \begin{cases} \dfrac{1-\beta}{1-\beta_T}\, p_1(n|d_1) & n < N \\[2ex] 1 - \dfrac{1-\beta}{1-\beta_T} \displaystyle\sum_{k=1}^{N-1} p_1(k|d_1) & n = N \end{cases}$$

6

$$p_o(n|d_o;N) = \begin{cases} \dfrac{(1-\alpha)(1-\beta)\, p_o(n) - \alpha(1-\alpha)\, p_1(n)}{(1-\alpha_T)(1-\alpha-\beta)} & n<N \\[2ex] 1 - \dfrac{(1-\alpha)(1-\beta) \sum_{k=1}^{N-1} p_o(k) - \alpha(1-\alpha) \sum_{k=1}^{N-1} p_1(k)}{(1-\alpha_T)(1-\alpha-\beta)} & n=N \end{cases}$$

$$p_1(n|d_o;N) = \begin{cases} \dfrac{\beta(1-\beta)\, p_o(n) - \alpha\beta\, p_1(n)}{\beta_T(1-\alpha-\beta)} & n<N \\[2ex] 1 - \dfrac{\beta(1-\beta) \sum_{k=1}^{N-1} p_o(k) - \alpha\beta \sum_{k=1}^{N-1} p_1(k)}{\beta_T(1-\alpha-\beta)} & n=N \end{cases}$$

$$p_o(n|d_1;N) = \begin{cases} \dfrac{\alpha(1-\alpha)\, p_1(n) - \alpha\beta\, p_o(n)}{\alpha_T(1-\alpha-\beta)} & n<N \\[2ex] 1 - \dfrac{\alpha(1-\alpha) \sum_{k=1}^{N-1} p_1(k) - \alpha\beta \sum_{k=1}^{N-1} p_o(k)}{\alpha_T(1-\alpha-\beta)} & n=N \end{cases}$$

$$p_1(n|d_1;N) = \begin{cases} \dfrac{(1-\beta)(1-\alpha)\, p_1(n) - (1-\beta)\beta\, p_o(n)}{(1-\beta_T)(1-\alpha-\beta)} & n<N \\[2ex] 1 - \dfrac{(1-\beta)(1-\alpha) \sum_{k=1}^{N-1} p_1(k) - (1-\beta)\beta \sum_{k=1}^{N-1} p_o(k)}{(1-\beta_T)(1-\alpha-\beta)} & n=N \end{cases}$$

$$
\text{7}\quad
\begin{cases}
p_o(n|T) = \begin{cases} p_o(n) & n < N \\ 1 - \sum\limits_{k=1}^{T-1} p_o(k) & n = N \end{cases} \\[2ex]
p_1(n|T) = \begin{cases} p_1(n) & n < N \\ 1 - \sum\limits_{k=1}^{T-1} p_1(k) & n = N \end{cases}
\end{cases}
$$

$$
\text{8}\quad
\begin{cases}
p_o(n) = \begin{cases} (1-\alpha_T)\, p_o(n|d_o;N) + \alpha_T\, p_o(n|d_1;N) & n < N \\ p_o(N|N) & n = N \end{cases} \\[2ex]
p_1(n) = \begin{cases} \beta_T\, p_1(n|d_o;N) + (1-\beta_T)\, p_1(n|d_1;N) & n < N \\ p_1(N|N) & n = N \end{cases}
\end{cases}
$$

$$
\text{9}\quad
\begin{cases}
\dfrac{1-\alpha_T}{1-\alpha} = \begin{cases} \dfrac{p_o(n|d_o)}{p_o(n|d_o;N)} & n < N \\[2ex] \dfrac{\sum\limits_{k=1}^{N-1} p_o(k|d_o)}{1-p_o(N|d_o;N)} & n = N \end{cases} \\[4ex]
\dfrac{\alpha_T}{\alpha} = \begin{cases} \dfrac{p_o(n|d_1)}{p_o(n|d_1;N)} & n < N \\[2ex] \dfrac{\sum\limits_{k=1}^{N-1} p_o(k|d_1)}{1-p_o(N|d_1;N)} & n = N \end{cases}
\end{cases}
$$

9 (cont.)

$$\frac{\beta_T}{\beta} = \begin{cases} \dfrac{p_1(n|d_o)}{p_1(n|d_o;N)} & n < N \\[2ex] \dfrac{\sum\limits_{k=1}^{N-1} p_1(n|d_o)}{1-p_1(N|d_o;N)} & n = N \end{cases}$$

$$\frac{1-\beta_T}{1-\beta} = \begin{cases} \dfrac{p_1(n|d_1)}{p_1(n|d_1;N)} & n < N \\[2ex] \dfrac{\sum\limits_{k=1}^{N-1} p_1(k|d_1)}{1-p_1(N|d_1;N)} & n = N \end{cases}$$

10

$$\frac{1-\alpha_T}{1-\alpha} \cdot \frac{\beta}{\beta_T} = \begin{cases} \dfrac{p_1(n|d_o;N)}{p_o(n|d_o;N)} & n < N \\[2ex] \dfrac{1-p_1(N|d_o;N)}{1-p_o(N|d_o;N)} & n = N \end{cases}$$

$$\frac{1-\beta}{1-\beta_T} \cdot \frac{\alpha_T}{\alpha} = \begin{cases} \dfrac{p_1(n|d_1;N)}{p_o(n|d_1;N)} & n < N \\[2ex] \dfrac{1-p_1(N|d_1,N)}{1-p_o(N|d_1;N)} & n = N \end{cases}$$

## 9. COMPUTER SIMULATION EXPERIMENTATION

The previous analysis applies to sequential detection on a continuous process where the problem of excess over the boundary does not arise. In practice observations are taken at discrete instances of time. Such a sampling procedure will generally cause the value of the test statistic to exceed the boundaries, thereby over-deciding on the hypothesis indicated. Thus a fraction of the last observation would have been sufficient to arrive at the same decision. Such fractional steps or observations are clearly impossible in a discrete sampling process. As a result of this the discrete test lasts longer than if the sampling were done continuously. The compensating effect is that the test statistic exceeding the boundary over-decides, thereby indicating the decision with a smaller probability of error than would have been obtained had the test statistics just touched the boundary. Consequently for a discrete test the boundaries can be set closer to zero to obtain the same performance ($\alpha, \beta$) as in the corresponding continuous test. Both of these factors must be simultaneously considered to obtain a valid measure of the effect of discrete sampling on the performance of the test.

In order to study the performance of discrete tests, the procedure of sequential testing was simulated on a digital computer and the pertinent test parameters and distributions were tabulated. Before we present and discuss the results of the computer analysis, we present a description of the simulation procedure.

For purposes of computer experimentation a more general procedure than that given by (3.5) can be studied. In particular we consider a Gaussian process with an unknown mean, but where the samples, $x_i$, are correlated. The process under study is the first order Markoff process, or "RC noise", where the correlation between the $i^{th}$ and $j^{th}$ observation is $\rho^{|i-j|}$. The test statistic for this case can be obtained from the likelihood

ratio (3.1). The result is[7]

$$Z_n = -\frac{1}{2}\,\frac{1-\rho}{1+\rho}\sum_{i=1}^{n}(a_1^2 - 2a_1x_i) - \frac{\rho}{1-\rho}\,[a_1^2 - a_1(x_1+x_n)] \qquad (9.1)$$

It can be easily verified that once $Z_k$ has been formed, $Z_{k+1}$ can be formed from the recurrence relation

$$Z_{k+1} = Z_k + a_1[(x_{k+1} - \rho x_k - \tfrac{1}{2}a_1(1-\rho)\;]/(1+\rho) \qquad (9.2)$$

in which the observed variable $x_1$ is formed from

$$x_k = a + n_k \qquad (9.3)$$

where $n_k$, representing the random component, is a Gaussian random variable having zero mean and unit variance and where $\underline{a}$ represents the true value of the parameter. On the first step $x_1 = a + n_1$. Correspondingly we have

$$Z_1 = a_1(x_1 - \frac{a_1}{2})$$

On the second and succeeding steps the additive noise is formed from the expression

$$n_{i+1} = w_i\sqrt{1-\rho^2} + n_i\rho \qquad i = 1,2,\ldots \qquad (9.4)$$

where $w_i$ is a random number from the same distribution as $n_i$ and $\rho$ is the coefficient of correlation. It can be easily verified that the autocorrelation of $n_i$ and $n_j$ where $n_i$ is constructed according to (9.4) is $\rho^{|i-j|}$. Fig. 9-1 illustrates the block diagram of the implementation of the detector represented by Eq. (9.1).

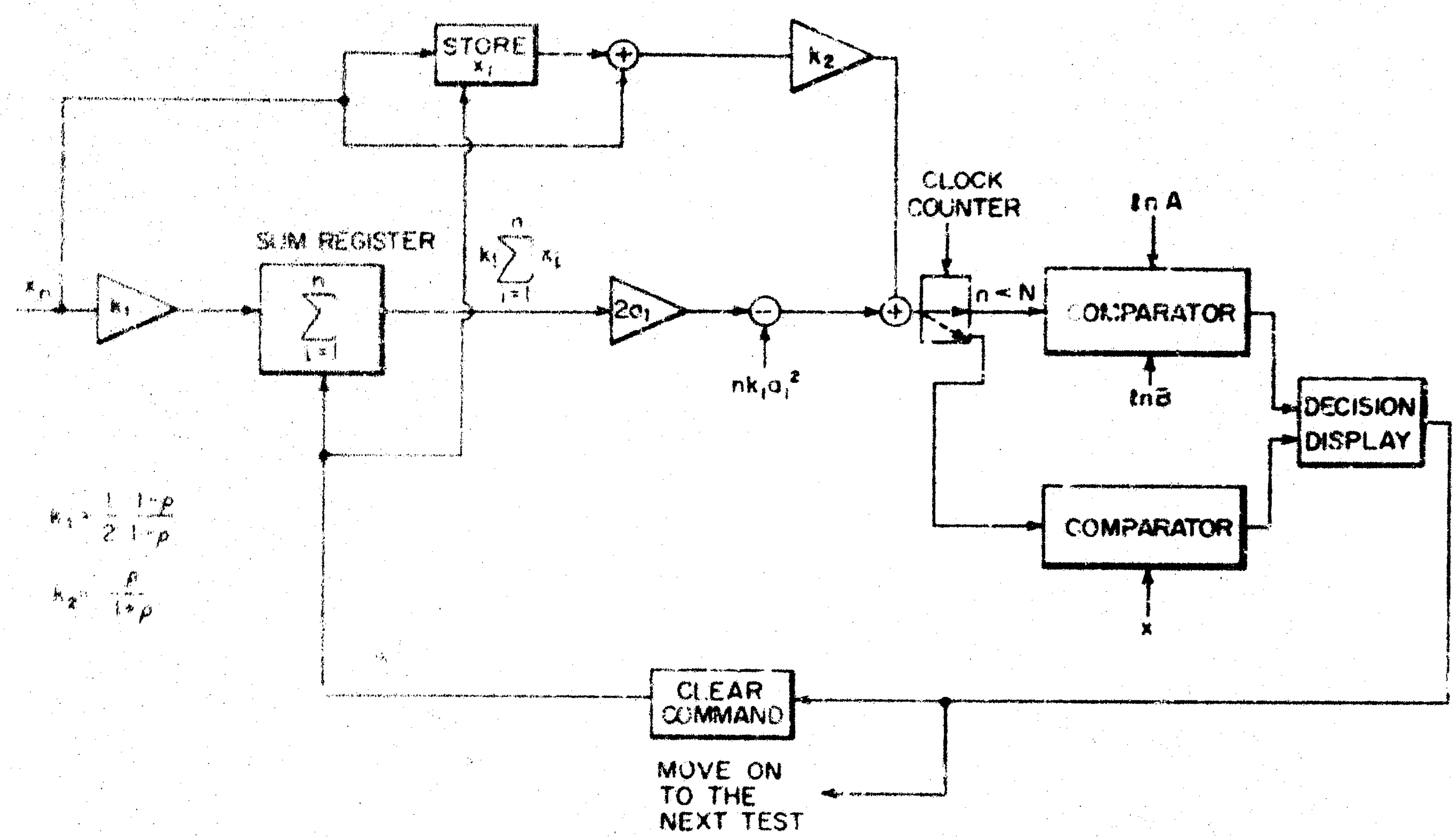


FIG. 9-1 BLOCK DIAGRAM FOR TRUNCATED SEQUENTIAL DETECTION OF CORRELATED SIGNALS

In running computer experiments for sequential tests, it is best to intentionally choose reasonably large values of the probabilities of false alarm and detection in order that the sample size (and thereby computer running time) be within reasonable limits. It is important to observe that if $\alpha$ and $\beta$ are small the proportion of trials ending in the incorrect decision will be correspondingly small and it will be difficult to compile enough data to achieve stable and reliable results about system performance. By virtue of the Bussgang-Marcus theorem it is possible, however, to study experimentally the conditional density of the sample size of the tests terminating in the more "popular" hypothesis and deduce from it the conditional density of the sample size of tests terminating in the other hypothesis. In Fig. 9-2 we show the result of 40,000 experiments. It is clear that with signal absent and $\alpha = 0.01$ most events terminate at the lower boundary and among 40,000 experiments only some 400 will terminate at the upper boundary which is insufficient to provide a stable picture of $p_o(n|d_1)$.

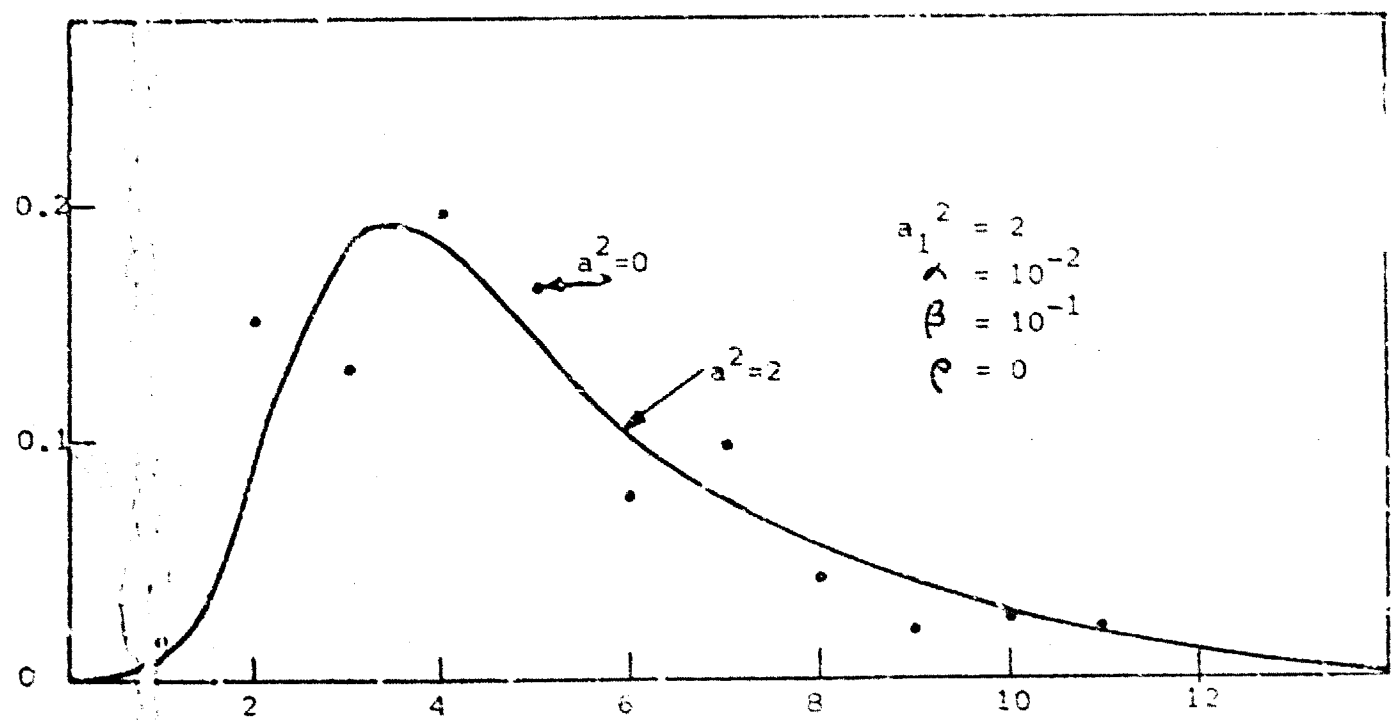


FIGURE 9-2 Conditional Probability Density of the Sample Size. Termination at the Upper Boundary: Discrete Process

## 10. DISCUSSION OF ILLUSTRATIVE EXAMPLES

The expressions for the ASN and OCF for a TST, (A.8) and (A.6), are quite involved and difficult to interpret. In order to gain insight into the interrelationship of these expressions, graphical results obtained by computer evaluation of these expressions are presented in Figs. 10-1 through 10-9 for selected values of the parameters. Figures 10-1 and 10-2 show the ASN as a function of the ratio $a/a_1$ for various values of the truncation parameter $a_1^2 T$. The uppermost curve, labeled $a_1^2 T = \infty$ gives the ASN for an untruncated test and the values of $\alpha$ and $\beta$ shown, which determine the fixed boundaries for all the tests, are for the untruncated case. The resulting values of $\alpha_T$ and $\beta_T$ for $\alpha = 0.01$ and $\beta = 0.1$ can be obtained from Fig. 10-3 since at $a=0$, OCF $= 1-\alpha$ and $a=a_1$, OCF $= \beta$. The complete performance of the test can thus be obtained by examining the two Figures 10-1 and 10-3. For example, if we truncate at $a_1^2 T = 5$, then $\alpha_T = 0.06$, $\beta_T = 0.27$ with $a_1^2$ASN $= 4.4$ when $a/a_1=0$.

Several comments can be made from Figures 10-1 and 10-2: The value of $a_1^2$ASN, as is expected, never exceeds the value $a_1^2 T$, but is very close to it when $a_1^2 T$ is small since many tests are then resolved only at truncation.

Also, we would expect, when $\alpha<\beta$ (i.e., $\ln A > |\ln B|$), that as $a_1^2 T$ increases, the value of $a/a_1$ for which $a_1^2$ASN is a maximum would decrease. That is, for small $a_1^2 T$ we would expect the drift of the Wiener process to require a larger positive slope to yield the maximum ASN than when $a_1^2 T$ is relatively large. This expectation is borne out by the figures, from which it is clear that as $a_1^2 T$ increases, the value of $a/a_1$ for which $a_1^2$ASN is a maximum decreases.

On closer examination we would expect that, when $\alpha<\beta$, the maximum value of $a_1^2$ASN occur for that value of $a/a_1$ for which the Wiener process has drifted slightly above the point $\frac{1}{2}(\ln A + \ln B)$ at the truncation stage $a_1^2 T$. That the maximum should occur for a value of $a/a_1$ for which the expected value

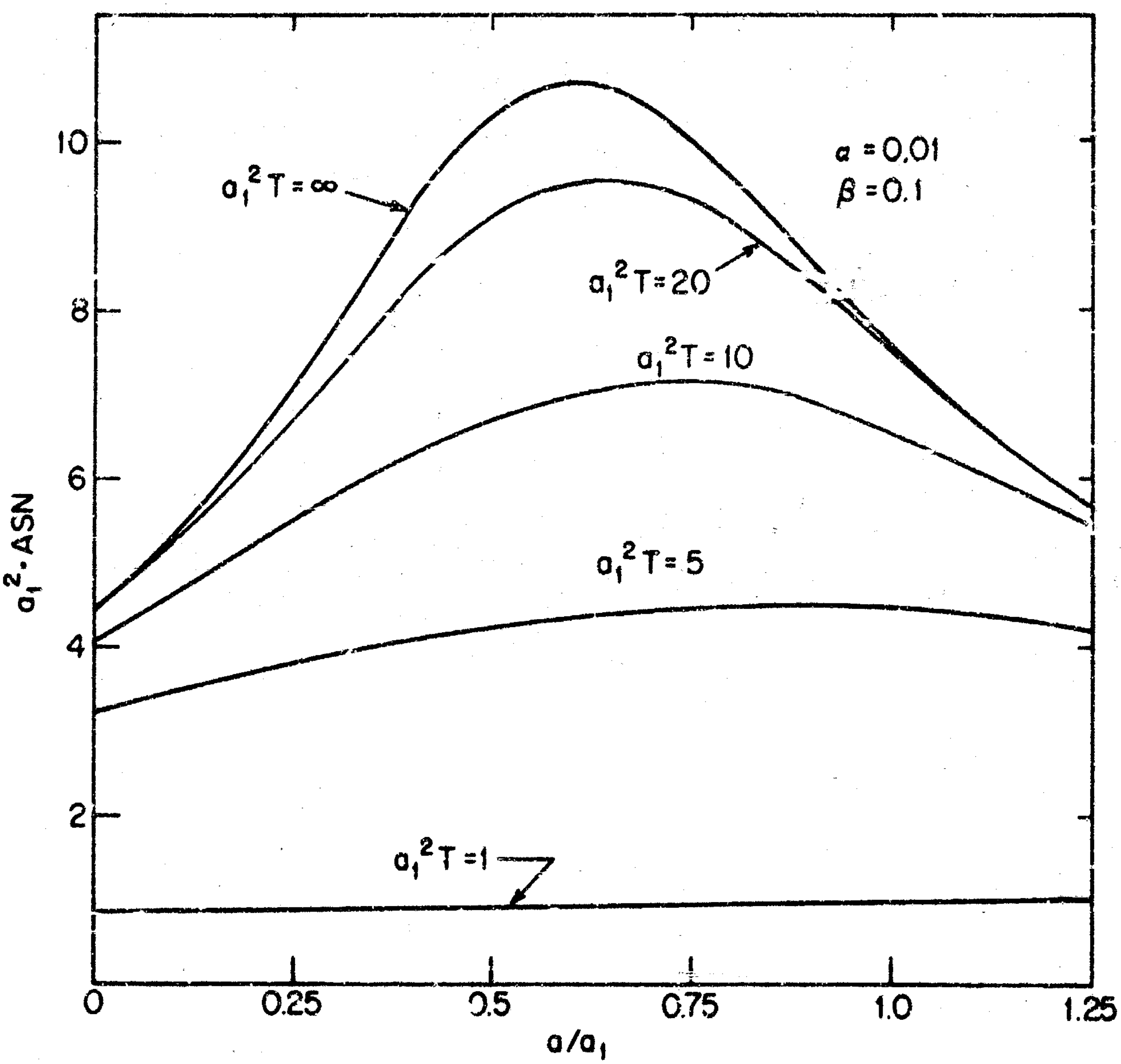


FIG. 10-1 ASN OF A TRUNCATED TEST vs. $a/a_1$; CONTINUOUS PROCESS

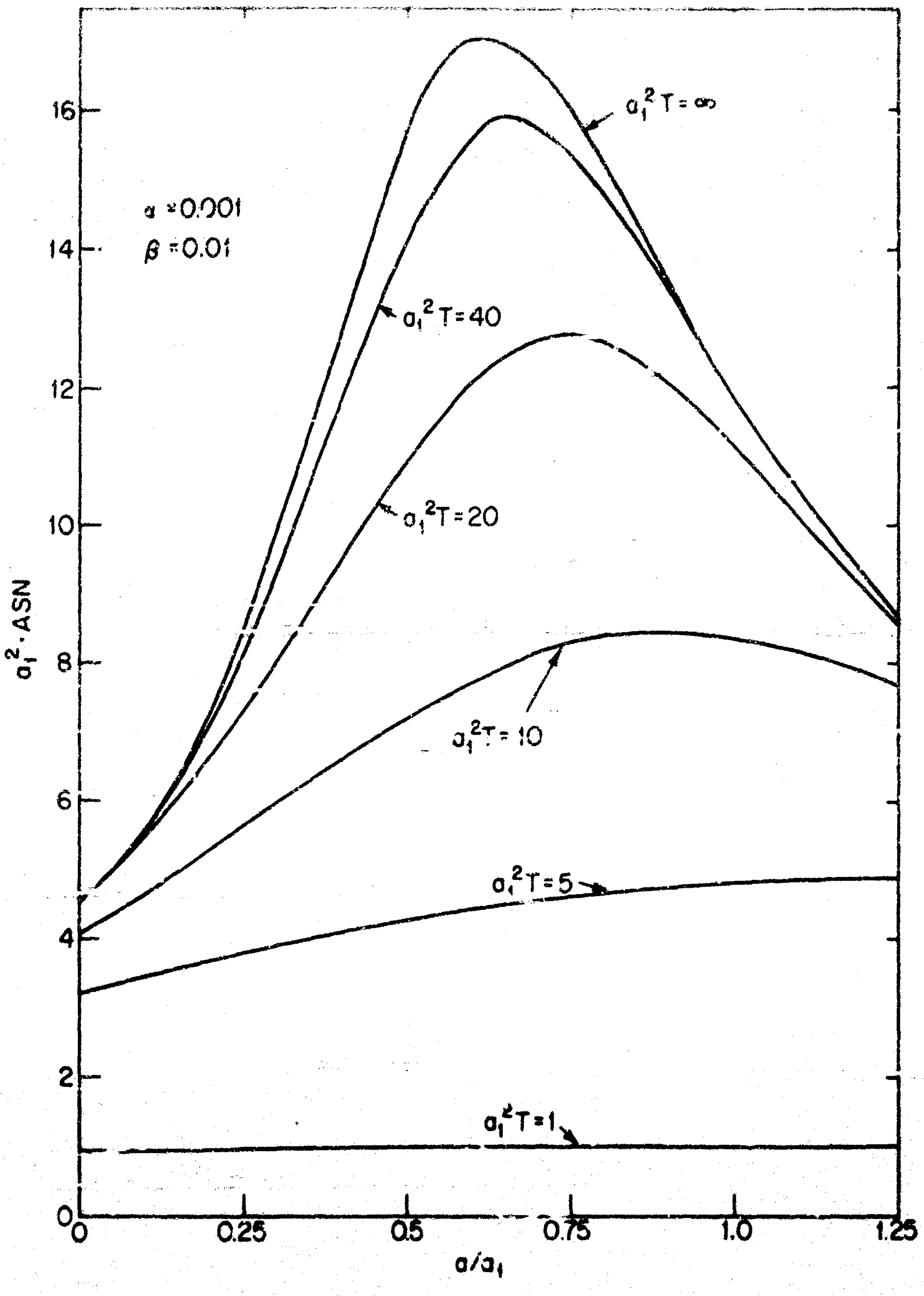


FIG. 10-2 ASN OF A TRUNCATED TEST vs $a/a_1$; CONTINUOUS PROCESS

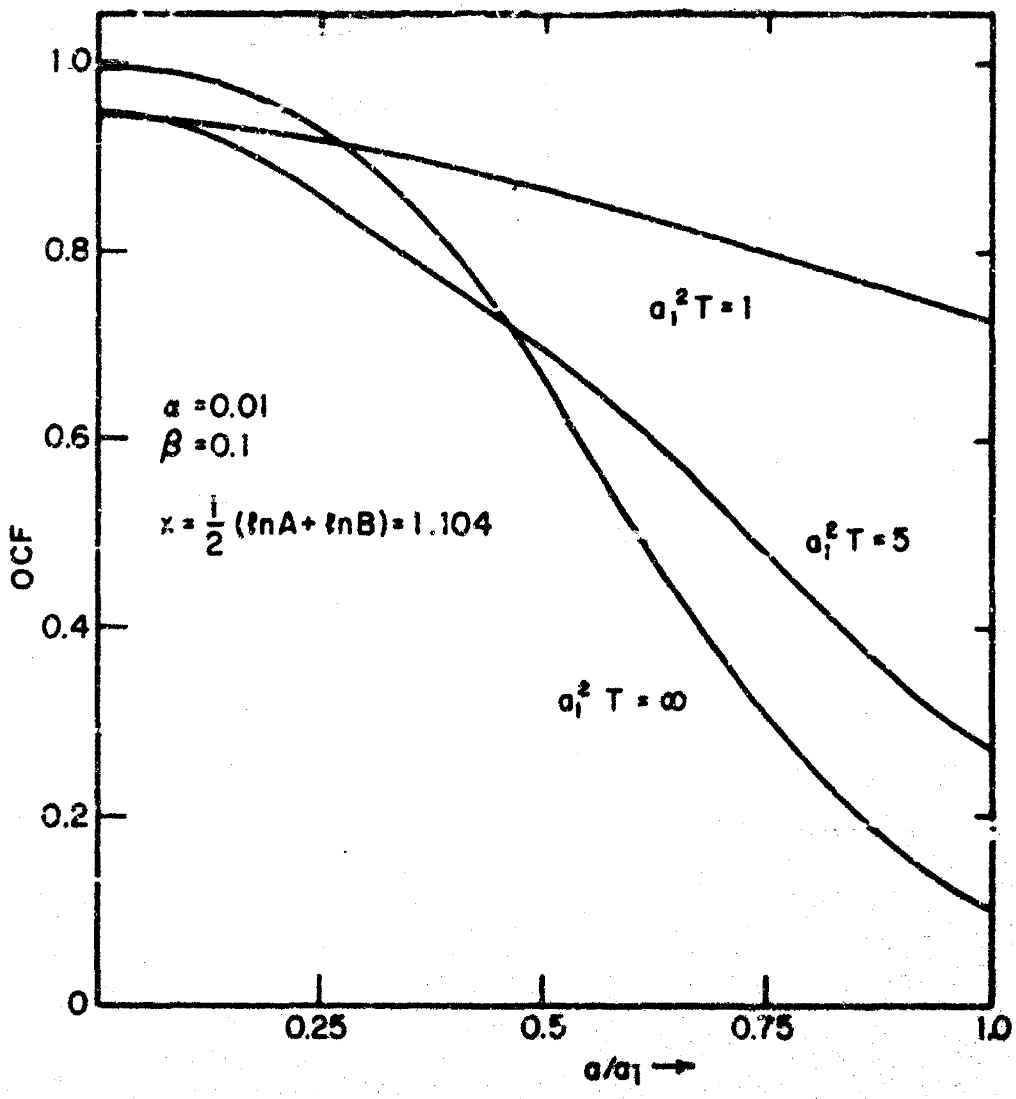


FIG. 10-3 OCF OF A TRUNCATED TEST vs $a/a_1$,
CONTINUOUS PROCESS

of the test statistic is somewhat larger than a point midway between the thresholds at the truncation point follows from the fact that for small values of t, $E(Z(t))$ is very close to the lower threshold as a result of which the tests would tend to terminate early unless the Wiener process drifted away from this absorbing boundary. The requirement that the process drift above the point $\frac{1}{2}(\ln A+\ln B)$ at $t=T$ to yield a maximum $a_1^2$ASN when $\alpha<\beta$ can be stated equivalently in terms of the ratio $a/a_1$. Thus, we want

$$E(Z(T)) > \frac{1}{2}(\ln A + \ln B)$$

which, from (A.4) leads to the inequality

$$a/a_1 > \frac{1}{2}\left(1 + \frac{\ln A + \ln B}{a_1^2 T}\right) \qquad (10.1)$$

The results in Figs. 10-1 and 10-2 show that the value of $a/a_1$ for which $a_1^2$ASN is a maximum does indeed satisfy inequality (10.1).

In Figure 10-4 is shown the probability density of the sample size for $a/a_1=0$ and $a/a_1=1$. The decision thresholds were set by $\alpha=0.01$ and $\beta=0.1$. These curves, calculated for the testing of the drift of a continuous Wiener process should be compared with the corresponding curves for the discrete detection procedure given by Figure 10-5. The similarity of the shape of the curves on the last two figures is evident. The main difference lies in that the mea[illegible]ed to terminate the discrete detection procedure is larger than in the continuous case, as is to be expected because in the case of continuous procedure some energy is lost in the excess over the boundaries. As discussed in Section 8 these densities hold for TST up to the truncation stage.

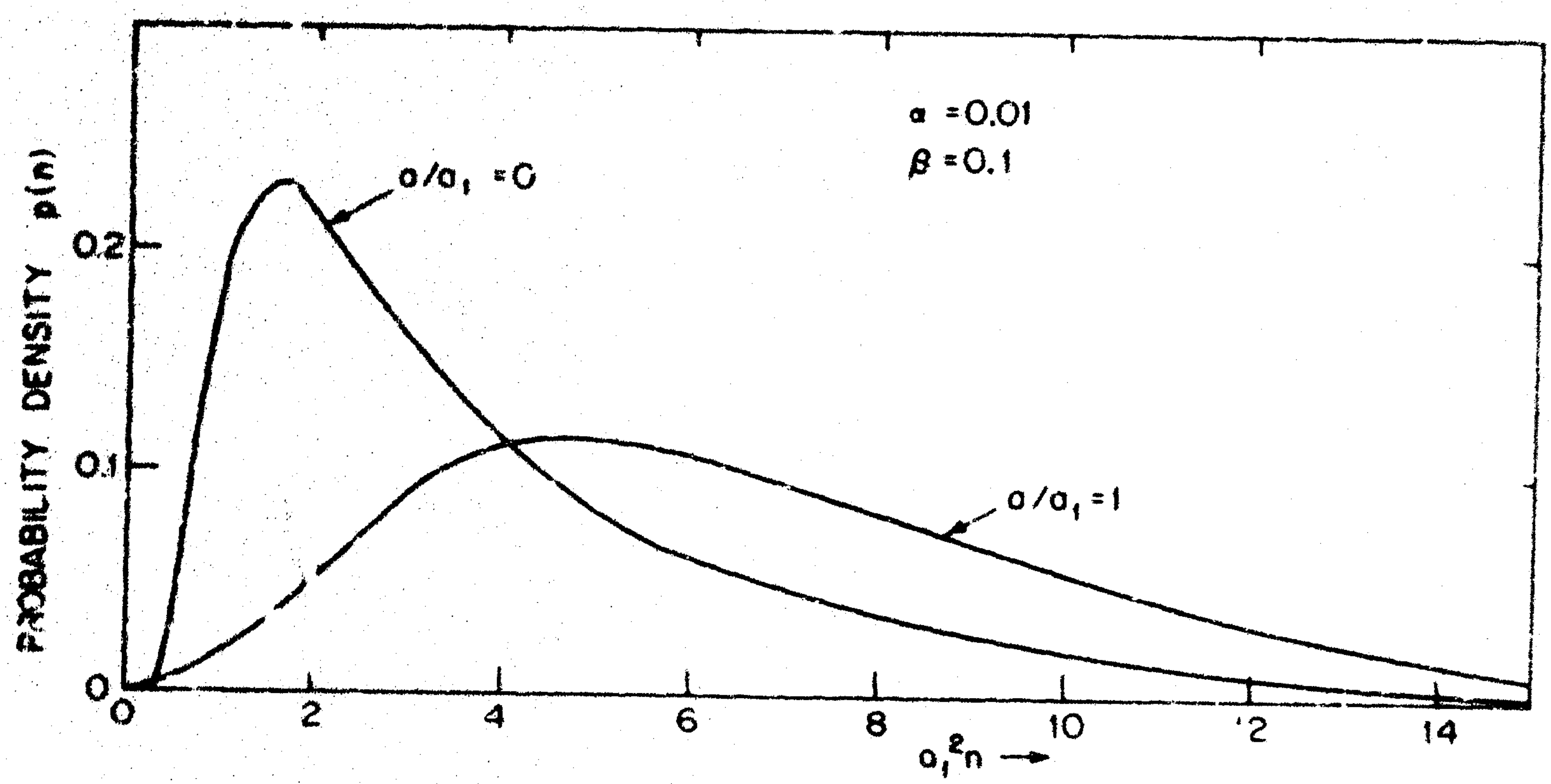


FIG 10-4 PROBABILITY DENSITY OF THE SAMPLE SIZE; CONTINUOUS PROCESS

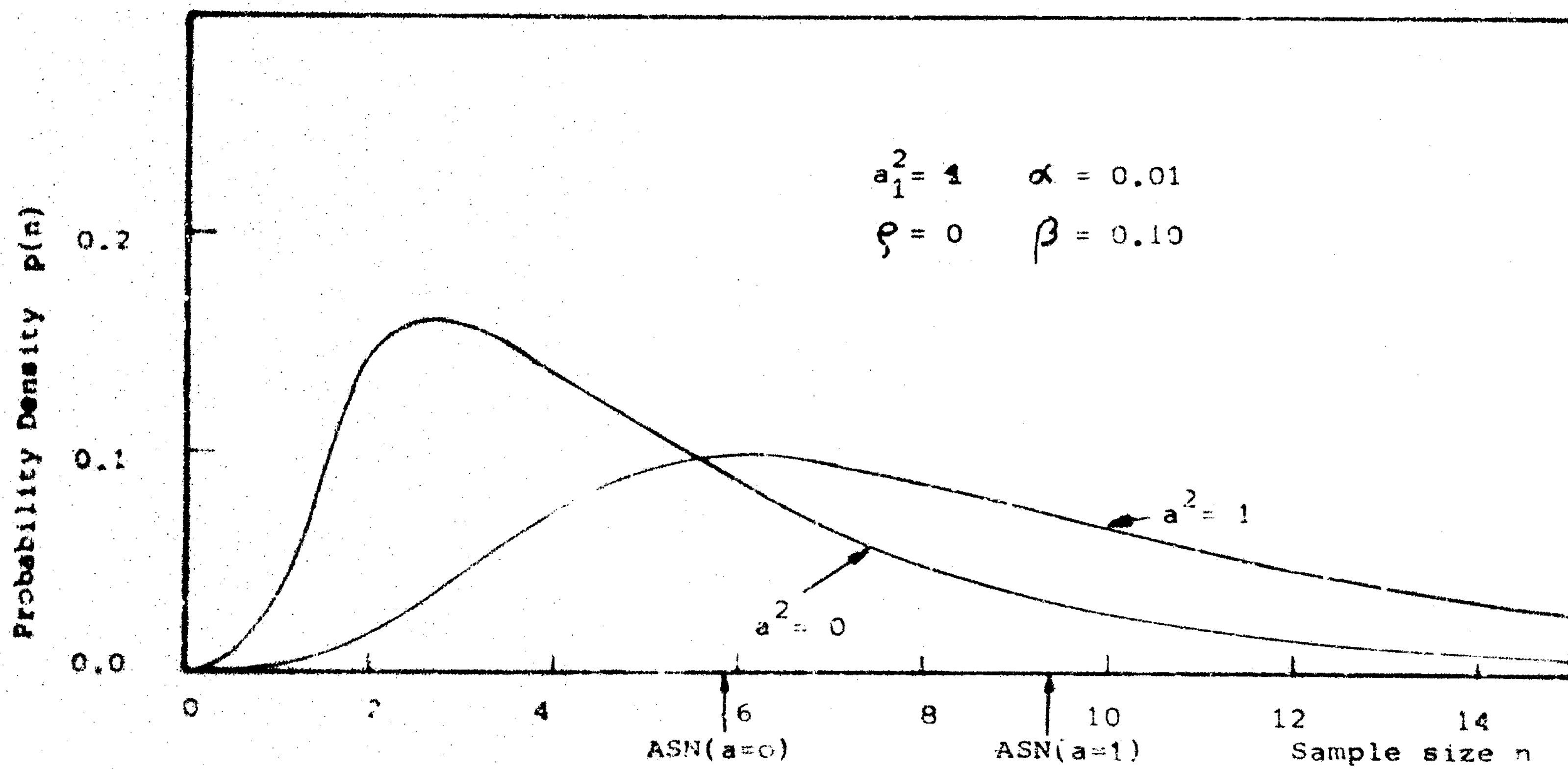


FIG. 10-5 PROBABILITY DENSITY OF THE SAMPLE SIZE : DISCRETE PROCESS

Curves for the OCF of a TST in Fig. 10-3 (as well as for $a_1^2T=\infty$) were shown with the terminal threshold set at the mid-point. When the terminal threshold is at the upper boundary or at $x=0$, as suggested by Wald, the resulting probabilities of error are shown in Fig. 10-6. As $x$ is increased from $x=0$ to $x=\ln A$, $\alpha_T$ decreases and $\beta_T$ increases, but the value of $x$ has a diminishing effect as the truncation stage increases. This influence of the terminal threshold can be studies also in Fig. 10-7. Further insight into relationships between the different quantities of the TST can be obtained from Figs. 10-8 and 10-9 in which ASN is plotted as a function of the truncation energy $a_1^2T$.

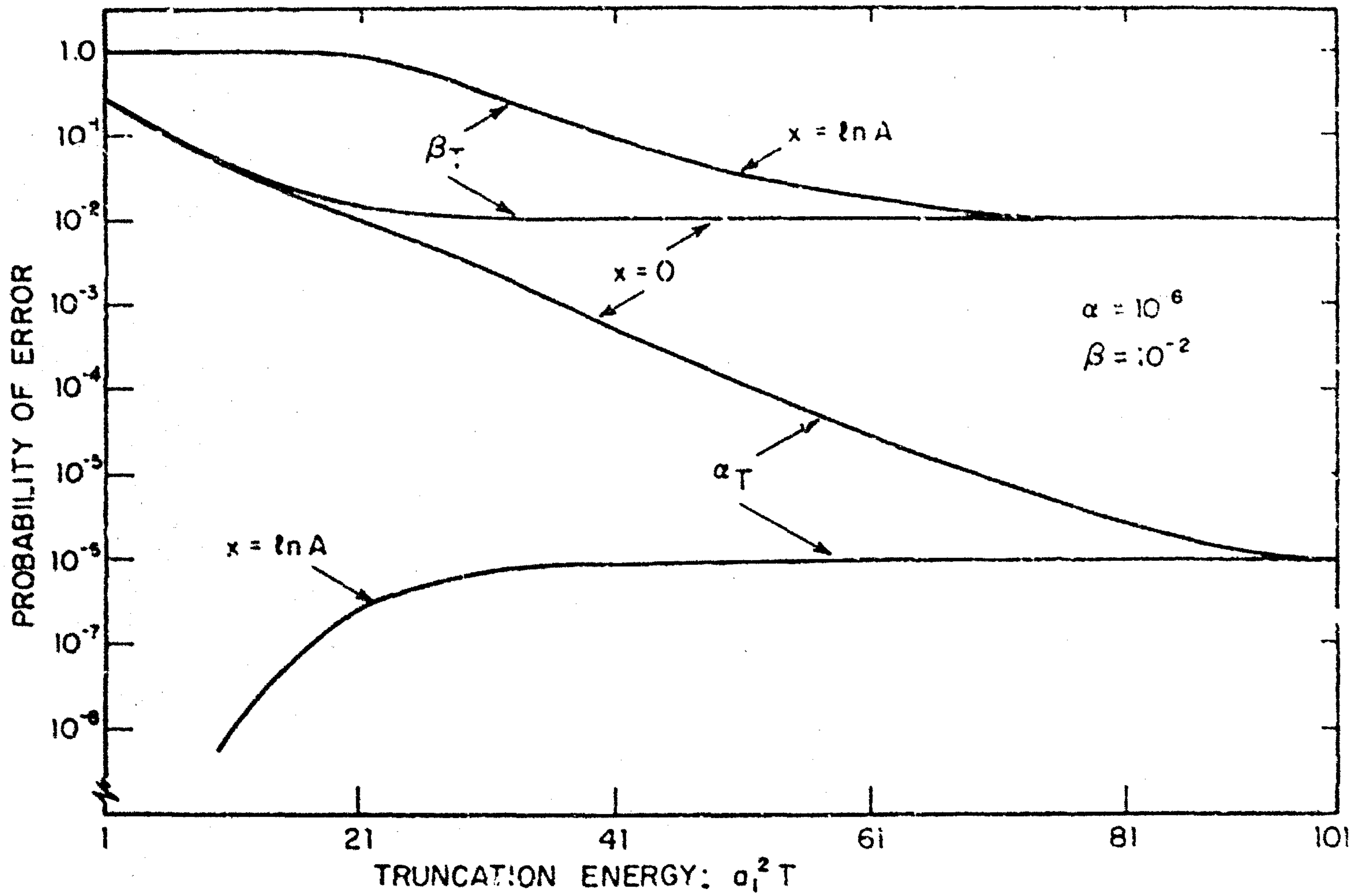


FIG. 10-6 PROBABILITIES OF ERRORS FOR A TST vs TRUNCATION STAGE, THRESHOLDS: $x = 0$, $\ell n A$; CONTINUOUS PROCESS

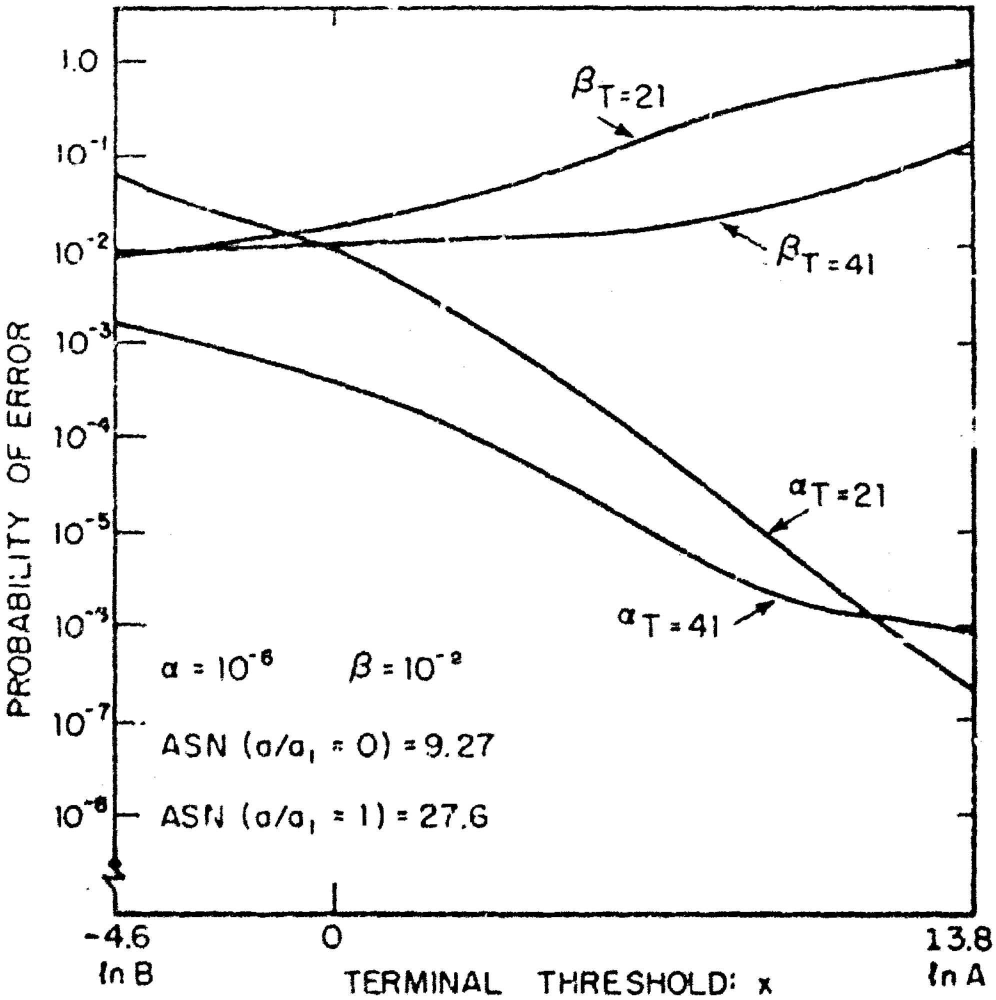


FIG. 10-7 PROBABILITIES OF ERROR FOR A TST vs. TERMINAL THRESHOLD, TRUNCATION STAGES: $a_1^2T = 21$, $a_1^2T = 41$; CONTINUOUS PROCESS

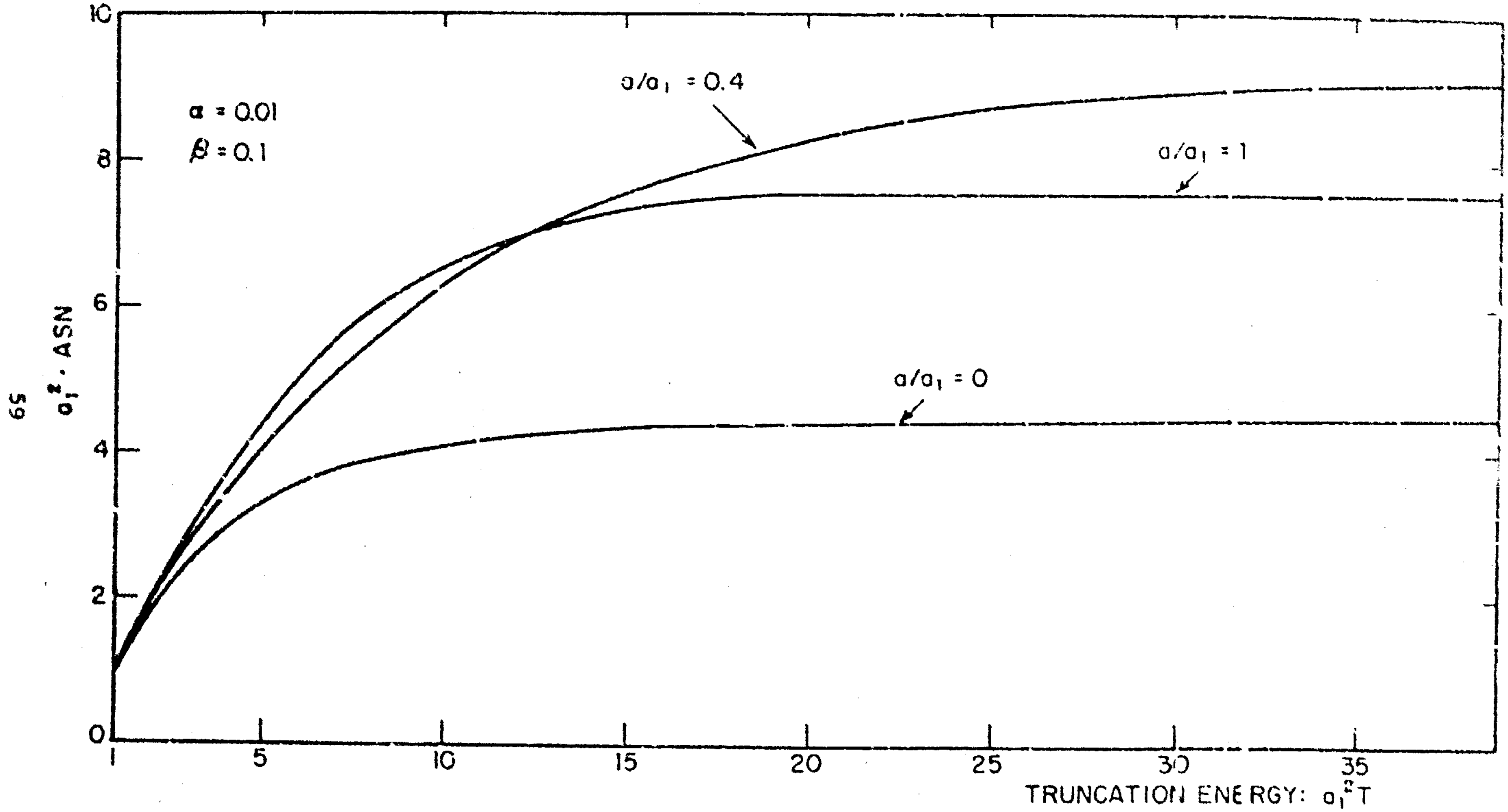


FIG. 10-8 AVERAGE SAMPLE OF A TST vs TRUNCATION ENERGY; CONTINUOUS PROCESS

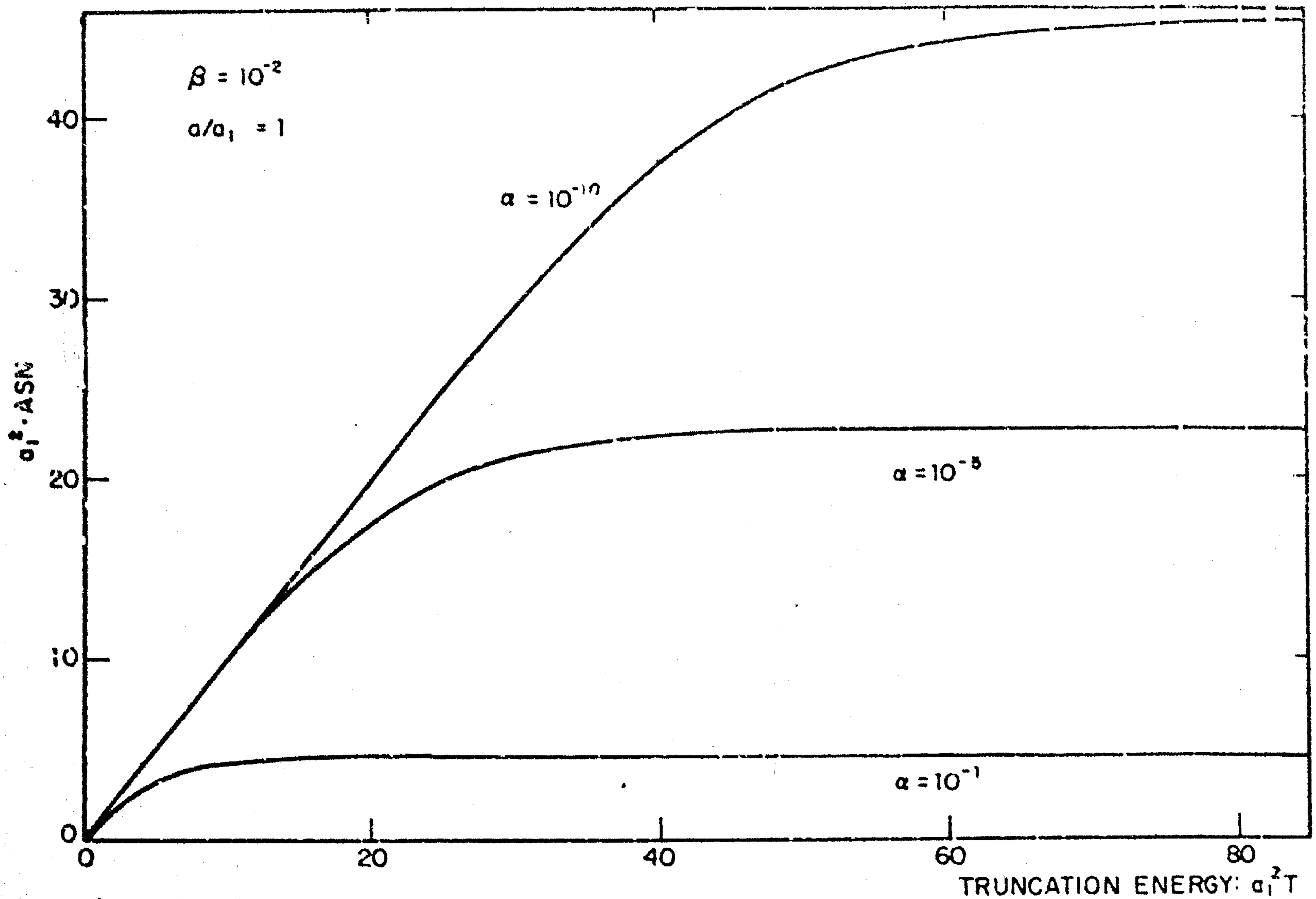


FIG. 10-9 AVERAGE SAMPLE NUMBER vs. TRUNCATION ENERGY; CONTINUOUS PROCESS

APPENDIX A

EXPRESSIONS FOR THE ASN, OCF, AND DISTRIBUTION FUNCTIONS

OF A TRUNCATED SEQUENTIAL TEST

## A.1 Introduction

In this Appendix we give exact expressions for the operating characteristic function, average sample number and distribution functions arising in sequentially testing for the mean of a Gaussian distribution. As in the main body of this report, we are interested in testing the hypothesis $H_0$ that the mean of the distribution is $a = 0$ against the alternative hypothesis $H_1$ that $a=a_1>0$. The samples $x_i$ used in arriving at the decision have a Gaussian distribution with mean a and unit variance. Thus the density function of the samples is given by

$$f(x_i,a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-a)^2}{2}} \tag{A.1}$$

The test statistic at the $n^{th}$ stage is given by

$$Z_n = \sum_{k=1}^{n} z_k \tag{A.2}$$

where $z_k$ is obtained from the $k^{th}$ sample $x_k$ by means of logarithm of the likelihood ratio

$$z_k = \ln \frac{f(x_k, a_1)}{f(x_k, 0)} = - \frac{a_1^2}{2} (1 - \frac{2x_k}{a_1}) \tag{A.3}$$

and where the expected value of $z_k$ is $E(z_k) = \mu = -\frac{a_1^2}{2}(1 - \frac{2a}{a_1})$.

The test statistic $Z_n$ is compared against two parallel thresholds $\ln A = \ln \frac{1-\beta}{\alpha}$ and $\ln B = \ln \frac{\beta}{1-\alpha}$ such that decision $d_o$ to accept $H_o$ is made if $Z_n \leq \ln B$, whereas decision $d_1$ to accept $H_1$ is made if $Z_n \geq \ln A$. The testing procedure continues as long as $\ln B < Z_n < \ln A$. If at stage n=N no decision has been reached, hypothesis $H_1$ is accepted if $Z_N \geq x$, and $H_o$ is accepted if $Z_N < x$.

It is very difficult to determine the exact values of the operating characteristic function and average sample number of a sequential test as specified above. This difficulty is due mainly to the fact the test will not in general terminate with $Z_n$ equal exactly to lnA or to lnB. In general $Z_n$ will exceed these boundaries, and it is the effect of this excess which is difficult to analyze.

In order to avoid the difficulties of excess over the boundary when the test statistic is discrete, we replace $Z_n$ by a continuous test statistic $Z(t)$ on which the same procedure is followed as in the discrete case. We are, then, testing for the drift of a Wiener process. This being the case, we have

$$E[Z(t)] = \mu t = -\frac{a_1^2}{2}(1 - \frac{2a}{a_1})t \tag{A.4}$$

$$E[Z(t) - E(Z(t))]^2 = a_1^2 t \tag{A.5}$$

and termination time T in place of $E\,Z_m = \mu m$, $E[Z(t)-E(Z(t))]^2 = a_1^2 m$ and termination time N.

### A.2 The Operating Characteristic Function

In a continuous sequential test the probability of false alarm will be exactly $\alpha$ and the probability of false dismissal will be exactly $\beta$, when the test is untruncated. In fact, the OCF and ASN for such a test can be written down directly. In the truncated case for a continuous sequential test, the situation is somewhat involved. The OCF is the probability that hypothesis $H_0$ is accepted. This is the probability that $Z(t)$ touches the boundary lnB before touching lnA, plus the probability that $Z(t)$ lies between the boundaries for $t<T$ and $Z(T)<x$. The expression for the general case of non-parallel boundaries is given by Anderson.[5] For our case we have

$$\mathrm{OCF} = \Phi(y) + \sum_{r=1}^{\infty} \Big[ e^{h(rc-\ln A)} \, \Phi\Big(-y - \frac{2(rc-\ln A)}{a_1\sqrt{T}}\Big)$$

$$-e^{hrc} \, \Phi\Big(-y - \frac{2rc}{a_1\sqrt{T}}\Big) - e^{-h(rc+\ln B)} \, \Phi\Big(y - \frac{2(rc+\ln B)}{a_1\sqrt{T}}\Big)$$

$$+ e^{-hrc} \, \Phi\Big(y - \frac{2rc}{a_1\sqrt{T}}\Big)\Big] \qquad \text{(A.6)}$$

where

$$A = (1-\beta)/\alpha, \quad B = \beta/(1-\alpha), \quad h = \Big(1-2\frac{a}{a_1}\Big),$$

$$y = \frac{x}{a_1\sqrt{T}} + \frac{a_1\sqrt{T}\,h}{2}, \quad \text{and} \quad c = \ln A - \ln B,$$

and where the function $\Phi(z)$ is given by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt \qquad (A.7)$$

It can be easily checked from Eqn. (A.6) that for $T = \infty$ $\alpha = 1 - OCF|_{a=0}$ and $\beta = OCF|_{a=a_1}$.

The approximations for $\alpha_T$ and $\beta_T$ given in Section 7 of the main body of this report are obtained from the general expression of the OCF given above.

A.3 The Average Sample Number

For the same truncated continuous sequential test as described above, the ASN is given by

$$a_1^2 \, ASN = (ASN)_1 + (ASN)_o + (a_1^2 T)P \qquad (A.8)$$

where

$$\begin{aligned}(ASN)_1 = \frac{2}{h}\sum_{r=0}^{\infty}\Bigg\{ & \Bigg[ e^{-h[rc+\ln A]}\, \Phi\left(\frac{a_1 h\sqrt{T}}{2} - \frac{(2rc+\ln A)}{a_1\sqrt{T}}\right) \\ & - e^{hrc}\,\Phi\left(-\frac{a_1\sqrt{T}\,h}{2} - \frac{2rc+\ln A}{a_1\sqrt{T}}\right)\Bigg][2rc+\ln A] \\ & - \Bigg[ e^{-h(r+1)c}\,\Phi\left(\frac{a_1 h\sqrt{T}}{2} - \left(\frac{2(r+1)c-\ln A}{a_1\sqrt{T}}\right)\right) \\ & - e^{h(rc-\ln B)}\,\Phi\left(-\frac{a_1 h\sqrt{T}}{2} - \frac{(2(r+1)c-\ln A)}{a_1\sqrt{T}}\right)\Bigg][2(r+1)c-\ln A]\Bigg\}\end{aligned} \qquad (A.9)$$

and $(ASN)_o$ is obtained when lnA and -lnB are interchanged and h is replaced by -h in (A.9). That is

$$(ASN(\ln A,\ \ln B,\ h))_0 = (ASN(-\ln B,\ -\ln A,\ -h))_1$$

Also $P = [1 - (P_1 + P_0)]$ where

$$P_1 = 1 - \Phi\left(\frac{a_1 h\sqrt{T}}{2} + \frac{\ln A}{a_1\sqrt{T}}\right) \tag{A.10}$$

$$+\sum_{r=1}^{\infty}\Big\{e^{-h(rc+\ln B)}\,\Phi\left(\frac{a_1 h\sqrt{T}}{2} - \frac{(2(r-1)c+\ln A)}{a_1\sqrt{T}}\right)$$

$$-e^{-rhc}\,\Phi\left(\frac{a_1 h\sqrt{T}}{2} - \frac{(2rc-\ln A)}{a_1\sqrt{T}}\right)$$

$$-e^{h(rc-\ln A)}\left[1-\Phi\left(\frac{a_1 h\sqrt{T}}{2} + \frac{2rc-\ln A}{a_1\sqrt{T}}\right)\right] + e^{rhc}\left[1-\Phi\left(\frac{a_1 h\sqrt{T}}{2} + \frac{2rc+\ln A}{a_1\sqrt{T}}\right)\right]\Big\}$$

and $P_0$ is obtained from $P_1$ with the same substitutions as prescribed for $(ASN)_0$.

It is not a difficult matter to check from Eqn. (A.8) that when $T = \infty$,

$$\tfrac{1}{2}\, a_1^2\ ASN\big|_{a=0} = -[(1-\alpha)\ln B + \alpha \ln A]$$

and

$$\tfrac{1}{2}\, a_1^2\ ASN\big|_{a=a_1} = \beta \ln B + (1-\beta)\ln A$$

These are the well-known expressions for the ASN for an untruncated test.

## A.4 The Density Function of the Sample Size

Again, for the same test which led to the expressions (A.6) for the OCF and (A.8) for the ASN, we get that the probability density p(t) of the test length t is given by

$$p(t) = p_1(t) + p_o(t) \tag{A.11}$$

where

$$p_1(t) = \frac{1}{a_1^3 t^{3/2}} \phi\left(\frac{a_1 h\sqrt{t}}{2} + \frac{\ln A}{a_1\sqrt{t}}\right) \sum_{r=o}^{\infty} \left\{(2rc + \ln A) e^{-\frac{2rc}{a_1^2 t}(rc+\ln A)} - (2c(r+1) - \ln A)\, e^{-\frac{2(r+1)c}{a_1^2 t}(rc-\ln B)} \right\} \tag{A.12}$$

and $p_o(t)$ is obtained from $p_1(t)$ by interchanging lnA with -lnB and replacing h by -h. The function $\phi(x)$ is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The function $p_i(t)$ is such that $P_i(t) = \int_{-\infty}^{t} p_i(x)dx$ is the probability that decision $d_i$ is made before time t. The expression $P_1(t)$ is of course the same as given by Eqn. (A.10), and has the property that $P_o(\infty) + P_1(\infty) = 1$. Also, for example, $P_1(\infty) = \alpha$ when a=0 and $P_o(\infty) = \beta$ when $a=a_1$.

## APPENDIX B

## APPROXIMATIONS TO THE DENSITY FUNCTION OF THE SAMPLE SIZE OF SEQUENTIAL TESTS

In the case of a sequential test for the mean of a Gaussian process, the exact expression for the density function p(t) of the sample size of an untruncated test can be obtained. This is equivalent to the distribution of the first passage time for a random walk with two parallel absorbing barriers. The exact expression is given by Eq. (A-11).

$$p(t) = p_1(t) + p_0(t) \qquad (B.1)$$

where

$$p_1(t) = \frac{1}{(a_1^2 t)^{3/2}}\, \phi\left(\frac{a_1 h\sqrt{t}}{2} + \frac{\ln A}{a_1\sqrt{t}}\right) \sum_{r=0}^{\infty} \left\{(2rc + \ln A)e^{-\frac{2rc}{a_1^2 t}(rc + \ln A)} - (2c(r+1) - \ln A)e^{-\frac{2(r+1)c}{a_1^2 t}(rc - \ln B)}\right\}$$

and $p_0(t)$ is obtained from $p_1(t)$ by interchanging $\ln A$ with $-\ln B$ and replacing h by -h. In this expression for the density function:

$c \quad = \ln A - \ln B,$

$h \quad = (1 - \frac{2a}{a_1})$ ,

a  true signal-to-noise ratio

$a_1$  preset signal-to-noise ratio

$\phi(x) = (\exp - x^2/2)/\sqrt{2\pi}$

For most purposes an approximate expression is adequate.

One such expression can be obtained by considering the first term of the infinite sum which appears in the exact expression. Thus we have that the probability density function p(t) of the test length t (or first passage time) can be approximated by

$$p(t) \doteq \frac{\left[\ell nA - (2c - \ell nA)B^{\frac{2c}{a_1^2 t}}\right]}{(a_1^2 t)^{3/2}} \phi\left(\frac{ha_1\sqrt{t}}{2} + \frac{\ell nA}{a_1\sqrt{t}}\right)$$

$$+ \frac{\left(-\ell nB - (2c + \ell nB)A^{-\frac{2c}{a_1^2 t}}\right)}{(a_1^2 t)^{3/2}} \phi\left(\frac{ha_1\sqrt{t}}{2} + \frac{\ell nB}{a_1\sqrt{t}}\right) \quad (B.2)$$

If the test proceeds by discrete sampling the approximate p.d.f. of the terminal stage n is obtained by replacing t with n.

A further simplification is possible if we also assume that in (B.2) B=0 and A > 0. The approximate expression for the p.d.f. then becomes

$$p(t) \doteq \frac{\ell nA}{\sqrt{2\pi}(\sigma^2 t)^{3/2}} e^{-\frac{1}{2\sigma^2}\left(\frac{\ell n^2 A}{t} + \mu^2 t - 2\mu \ell nA\right)} \quad (B.3)$$

where

$$\mu = -\frac{a_1^2}{2} + aa_1$$

and

$$\sigma^2 = a_1^2$$

The expression (B.3) checks after suitable substitutions, Wald's[2] formula (A.183, p.193).

A further approximation is possible if $(\mu \ell nA)/\sigma^2$ is large, i.e. if many steps are needed to reach the bound. The distribution of t given by (B.3) approaches then the Gaussian distribution about its mean

$$p(t) \doteq \frac{1}{(2\pi \ell nA\ \sigma^2/\mu^3)^{1/2}}\ e^{-\frac{1}{2\ell nA\ \sigma^2/\mu^3}(t-\ell nA/\mu)^2} \tag{B.4}$$

It is clear that the case A>0, B=0 represents essentially the random walk with only the upper boundary. The case of random walk with only the lower boundary is represented by B>0 and A infinite. The expressions appropriate to this case can be obtained simply by replacing $\ell nA$ by $-\ell nB$ wherever $\ell nA$ appears in (B.3) and (B.4).

Another useful approximation applies to the case A=1/B [see Bussgang and Middleton[4] (7.74), (7.75)]

We note also that if z is the logarithm of the probability ratio of a test with independent samples but is not a normal variable, then the p.d.f. of the sample size for B=0, A>0 can still be approximated by (B.3) where $\mu$ and $\sigma^2$ are now Ez and Var z.

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13. ABSTRACT


This monograph presents an exposition of the subject of truncated sequential tests. Although the main intent of this report is tutorial in nature, some previously unpublished results have been included.

Special emphasis has been given to the following topics: a) the practical necessity for truncation, b) the effects of truncation on the performance of the test, c) the results of computer simulation, and d) rules of truncation and application of principles of design.


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